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On the catalytic role of the phase-locked interaction of Tollmien–Schlichting waves in boundary-layer transition

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This paper is concerned with the nonlinear interaction between a planar and a pair of oblique Tollmien–Schlichting (T-S) waves which are phase-locked in that they travel with (nearly) the same phase speed. The evolution of such a disturbance is described using a high-Reynolds-number asymptotic approach in the so-called ‘upper–branch’ scaling regime. It follows that there exists a well-defined common critical layer (i.e. a thin region surrounding the level at which the basic flow velocity equals the phase speed of the waves to leading order) and the dominant interactions take place there. The disturbance is shown to evolve through several distinctive stages. In the first of these, the critical layer is in equilibrium and viscosity dominated. If a small mismatching exists in the phase speeds, the interaction between the planar and oblique waves leads directly to super-exponential growth/decay of the oblique modes. However, if the modes are perfectly phase-locked, the interaction in the first instance affects only the phase of the amplitude function of the oblique modes (so causing rapid wavelength shortening), while the modulus of the amplitude still evolves exponentially until the wavelength shortening produces a back reaction on the modulus (which then induces a super-exponential growth). Whether or not there is a small mismatch or a perfect match in the phase speeds, once the growth rate of the oblique modes becomes sufficiently large, the disturbance enters a second stage, in which the critical layer becomes both non-equilibrium and viscous in nature. The oblique modes continue to experience super-exponential growth, albeit of a different form from that in the previous stages, until the self-interaction between them, as well as their back effect on the planar mode, becomes important. At that point, the disturbance enters a third, fully interactive stage, during which the development of the disturbance is governed by the amplitude equations with the same nonlinear terms as previously derived for the phase-locked interaction of Rayleigh instability waves. The solution develops a singularity, leading to the final stage where the flow is governed by fully nonlinear three-dimensional inviscid triple-deck equations. The present work indicates that seeding a planar T-S wave can enhance the amplification of all oblique modes which share approximately its phase speed.

1. Introduction

Laminar–turbulent transition in an incompressible flat-plate boundary layer is often initiated by amplification of small-amplitude disturbances, known as Tollmien–Schlichting (T-S) waves. During the early stages of such transition, the disturbances are usually predominantly two-dimensional, and their initial development is described well by linear stability theory. However, sufficiently further downstream, three-dimensional disturbances are observed to amplify more rapidly and become dominant. This is in contrast to linear theory, which predicts that the fastest growing mode is two-dimensional. Observationally, a similar preferential growth three-dimensional disturbances occurs in the transition process in mixing layers and wakes, where transition is caused by inviscid Rayleigh instability waves. Since turbulence is characteristically three-dimensional, the preferential growth of three-dimensional disturbances is a crucial step in the transition process. Because of this, identifying the mechanisms which promote fast growth of such disturbances has been one of the key issues in transition study (Kachanov 1994).

One of a number of mechanisms that have been proposed is *subharmonic resonance*. The possible existence of such a resonant triad of modes, consisting of a planar and a pair of oblique modes, in the Blasius boundary layer was first suggested by Raetz (1959). Craik (1971) then derived evolution equations for the interacting instability modes by using weakly nonlinear theory in a heuristic fashion. His theoretical work prompted the landmark experiments by Kachanov & Levchenko (1984) that established the importance of subharmonic resonance in boundary-layer transition. Self-consistent asymptotic mathematical descriptions of resonant-triad interactions were constructed in a series of papers by Goldstein & Lee (1992), Goldstein (1994, 1995*a*), Mankbadi, Wu & Lee (1993) and Lee (1997), Wu (1993, 1995), Wundrow, Hultgren & Goldstein (1994), for boundary layers with zero, adverse or favourable pressure gradients. An important finding is that through a parametric resonance, the planar wave causes the subharmonic oblique waves to amplify super-exponentially (Goldstein & Lee 1992; Mankbadi *et al.* 1993). This key prediction has been confirmed experimentally for a decelerating boundary layer (Borodulin, Kachanov & Koptsev 2002*a*; Borodulin *et al.* 2002*b*).

An alternative explanation for the preferential growth of three-dimensional disturbances is *secondary instability* (e.g. Herbert 1988). In this approach, the original steady basic flow plus the dominant two-dimensional planar mode is taken to be the base flow, and the instability of this new base flow to three-dimensional disturbances is studied. Hence, in contrast to the asymptotic theories where the planar wave is allowed to evolve, in this approach the amplitude of the evolving planar wave is treated, in a somewhat *ad hoc* manner, as a quasi-steady parameter. It is found that when the amplitude just exceeds a (moderate) threshold, a three-dimensional wave with the subharmonic frequency of the planar mode becomes the most unstable mode (i.e. more unstable than the planar wave). However, if the amplitude of the planar mode is increased further, the most unstable (three-dimensional) disturbance reverts to having the fundamental frequency.

In both the above explanations, the dominant planar mode at first primarily enhances the growth of a narrow band of disturbances with frequencies centred at the subharmonic frequency. In experiments, however, occurrence of rapid growth is not restricted to such a narrow band. For instance, in the late stage of plane-wake transition, three-dimensional disturbances with a broadband of frequencies amplify to overtake the originally dominant planar mode (Corke, Krull & Ghassemi 1992; Williamson & Prasad 1993*a*, *b*). Prompted by this observation, Wu & Stewart (1996) proposed a new general mechanism, which they referred to as a *phase-locked interaction*. They showed that an effective nonlinear interaction could take place between planar and oblique Rayleigh instability waves which have the same phase speed (and hence are phase-locked). The interaction within their common critical layer generates a strong difference mode, which in turn interacts with the oblique mode to influence the development of the latter. While this mechanism is somewhat weaker than a resonant triad interaction, it operates under a much less restrictive condition (in that the subharmonic relation between the frequencies is not required). Yet this mechanism can significantly enhance the growth of the oblique wave and may eventually cause it to grow super-exponentially. In summary, the crucial difference from subharmonic resonance is that the phase-locked mechanism predicts that a dominant planar mode is able to preferentially excite all oblique modes sharing its phase speed.

In the present paper, we study the role of phase-locked interactions of T-S waves. Our investigation is motivated by the experimental observation that in boundary-layer transition, three-dimensional disturbances in a wide frequency range grow to a significant level, despite not satisfing the resonant-triad conditions (e.g. Corke & Mangano 1989). Numerical simulations have also revealed such a phenomenon (e.g. Spalart & Yang 1987).

As in many of the related previous theoretical studies of boundary-layer transition (Mankbadi *et al.* 1993; Wu 1993; Wu, Leib & Goldstein 1997), we shall focus on the *upper-branch scaling regime*, since (*a*) in most experiments on boundary-layer transition, nonlinear effects do not become noticeable until the upper branch of the neutral curve is approached (e.g. Klebanoff, Tidstrom & Sargent 1962; Kachanov & Levchenko 1984; Corke & Mangano 1989), and (*b*) the upper-branch regime covers almost the entire linearly unstable region, including the overlapping domain between the upper and lower branch regions, as pointed out by Goldstein & Durbin (1986). The choice of the upper-branch regime is crucial for the present investigation, because the notion of a phase-locked interaction is based on nonlinear interactions within a critical layer, and the critical layer is distinct only in this regime.

We note that the upper-branch asymptotic approximation for the linear growth rate is not accurate at moderately high Reynolds numbers (see e.g. Reid 1965; Healey 1994). However, we do not believe that this shortcoming constitutes a reason for developing nonlinear theory based on the upper-branch scaling. This is because the nonlinear growth rate of a disturbance depends principally on its modal shape, and the latter is predicted reasonably well by high-Reynolds-number asymptotic theory. Further, given that the nonlinear growth quickly overtakes the linear one, the qualitative behaviour of the disturbance should not be significantly affected by the relative inaccuracy of the linear growth rate. Thus, although the finite-Reynolds-number approach based on the Orr–Sommerfeld equation gives a good approximation to linear growth rate, we believe that the high-Reynolds-number formulation offers the best opportunity to construct a self-consistent theory that can at least qualitatively describe the major nonlinear effects.

The rest of the paper is organized as follows. In §2, the problem is formulated. We assume that planar and oblique T-S modes are present, propagating at the same phase speed to leading order, but allowing for a small degree of mismatching. As the modes propagate downstream they interact within their common critical layer, until at some point there is a nonlinear feedback; we consider this stage in §3. Specifically the mutual interaction is shown to induce super-exponential growth or decay of the oblique modes if the phase-speed difference is non-zero. A special case arises if the phase speeds are exactly the same, since the interaction in the first instance only alters the wavelength of the oblique modes, leaving their modulus to evolve exponentially (rather than super-exponentially). However, further downstream the wavelength alteration becomes rapid enough to produce a back reaction on the modulus causing, even in this special case, the latter to grow super-exponentially. In either case, once the oblique modes have acquired a sufficiently large growth rate, the disturbance enters a second stage, which is distinguished in that non-equilibrium effects become as important as viscous effects in the critical layer. The oblique modes then evolve over a faster scale than the planar wave. In particular, provided their initial magnitude is sufficiently small that their effect on the planar mode is negligible, the amplitude function of the oblique modes then takes a WKBJ form. This regime, which we shall refer to as the WKBJ stage, is considered in § 4. The continued rapid growth of the oblique modes eventually leads to a third, fully-coupled stage, when the self-interaction between the oblique modes, as well as their back effect on the planar wave, comes into play. This third stage is studied in § 5. By observing the similarity between the present problem and the related problem of Rayleigh waves (Wu & Stewart 1996), it is deduced that the amplitude equations in this stage are the same as those for the Rayleigh problem, but without the linear growth terms. The appropriate initial conditions are derived by matching with the WKBJ stage. The amplitude equations are solved numerically, and the solution is found to develop a finite-distance singularity, leading to a strongly nonlinear stage, governed by the unsteady inviscid triple-deck equations. This final stage is highlighted in § 6. Further, we show in the Appendix that all the weakly nonlinear stages through which the disturbance evolve are governed by appropriate limiting forms of the amplitude equations derived by Wu & Stewart (1996) for the nonlinear non-equilibrium viscous critical-layer regime. In § 7 we discuss the implication of this, and draw some general conclusions from the results of our analysis.

2. Formulation and linear solutions

The basic flow is taken to be the incompressible Blasius boundary layer on a semi-infinite flat plate. It is described in terms of Cartesian coordinates (x, y, z) with its origin at a point on the plate a (dimensional) distance L downstream from the leading edge, where x and y are along and normal to the plate, respectively, and z is the spanwise direction. The velocity components in these directions are denoted by u, v and w, respectively. The length, time, velocity and the pressure p are normalized by δ^*, δ^*/U_∞, U_∞ and ρU_∞^2, respectively, where U_∞ is the free-stream velocity, ρ is the fluid density, ν is the kinematic viscosity, and $\delta^* = (\nu L/U_\infty)^{1/2}$ is the boundary-layer thickness at $x = 0$. The local Reynolds number $R = U_\infty \delta^*/\nu$ is assumed to be large, i.e. $R \gg O(1)$. Near the wall, the basic-flow profile, U_B, has the behaviour

$$U_B \sim \lambda y + \lambda_4 y^4 + \cdots, \quad \text{as} \quad y \to 0, \tag{2.1}$$

where $\lambda_4 = -\lambda^2/48$. Strictly speaking, λ depends on the slow streamwise variable x/R, but to the order of approximation in the present paper it can be treated as constant with $\lambda \approx 0.332$.

We shall consider the interaction between a planar T-S wave with a dimensional frequency $\tilde{\Omega}$, and a pair of oblique T-S waves with a common dimensional frequency Ω. The upper-branch regime for the Blasius boundary layer corresponds to the scaling (Reid 1965; Bodonyi & Smith 1981)

$$\tilde{\Omega}\nu/U_\infty^2, \quad \Omega\nu/U_\infty^2 \sim R^{-6/5}. \tag{2.2}$$

It is convenient to introduce a small parameter

$$\sigma = R^{-1/10},$$

and to define scaled frequencies

$$\tilde{\omega} = \sigma^{-12}\big(\tilde{\Omega}\nu/U_\infty^2\big), \quad \omega = \sigma^{-12}\big(\Omega\nu/U_\infty^2\big).$$

Linear stability theory (e.g. Bodonyi & Smith 1981) suggests the introduction of scaled coordinates

$$\tilde{X} = \sigma\tilde{\alpha}x - \sigma^2\tilde{\omega}t, \quad X = \sigma\alpha x - \sigma^2\omega t, \quad Z = \sigma\beta z, \quad x_1 = \sigma^4 c^{-1}x, \tag{2.3}$$

where $\tilde{X}$, X and Z are the 'fast' variables describing the oscillation and the spanwise variation of the T-S waves, respectively, while x_1 is an appropriate 'slow' variable describing their growth. The parameter $\tilde{\alpha}$ denotes the scaled streamwise wavenumber of the planar mode, while α and β are the scaled streamwise and spanwise wavenumbers of the oblique modes, respectively. It follows from (2.3) that the multiple-scale substitutions hold

$$\frac{\partial}{\partial t} \to -\sigma^2\omega\frac{\partial}{\partial X} - \sigma^2\tilde{\omega}\frac{\partial}{\partial \tilde{X}}, \quad \frac{\partial}{\partial x} \to \sigma\alpha\frac{\partial}{\partial X} + \sigma\tilde{\alpha}\frac{\partial}{\partial \tilde{X}} + \sigma^4 c^{-1}\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial z} \to \sigma\beta\frac{\partial}{\partial Z}. \tag{2.4}$$

We assume that the phase speeds of the planar and oblique modes, $\tilde{c} \equiv \tilde{\omega}/\tilde{\alpha}$ and $c = \omega/\alpha$, are equal to the leading order, but are allowed to have an $O(\sigma^2)$ difference, i.e.

$$\tilde{c} = c + \sigma^2\Delta, \tag{2.5}$$

where $\Delta = O(1)$ is a parameter characterizing the phase-speed mismatch. The effect of phase-speed mismatch was considered by Wu (1996) in his work on the resonant triad and phase-locked interactions of sinuous and varicose modes in a plane wake, and in a closer context by Jennings (1997), who studied the phase-locked interaction of T-S waves in the high-frequency limit of the lower-branch regime. In Wu (1996), the phase-speed mismatch was treated as a special case of amplitude modulation.

The wavenumber α and the phase speed c expand in the form

$$\alpha = \alpha_0 + \sigma\alpha_1 + \sigma^2\alpha_2 + \sigma^3 \ln\sigma\,\alpha_{3L}, \tag{2.6}$$

$$c = \frac{\omega}{\alpha} = c_0 + \sigma c_1 + \cdots, \quad \text{with} \quad c_0 = \frac{\omega}{\alpha_0}, \quad \text{etc.}, \tag{2.7}$$

and a similar expansion holds for $\tilde{\alpha}$. However, we shall only need to retain the leading–order terms in the expansions as higher-order ones do not affect the nonlinear interactions considered in this paper. In terms of the above scaled variables, the disturbance in the main part of the boundary layer, to leading order, takes the form

$$\left.\begin{aligned}
u &= \epsilon B(x_1)\phi'(y)\,\mathrm{e}^{\mathrm{i}\tilde{X}} + \delta A(x_1)\bar{u}_1(y)\,\mathrm{e}^{\mathrm{i}X}\cos Z + \text{c.c.} + \cdots,\\
v &= -\epsilon\sigma\tilde{\alpha}_0\mathrm{i}B(x_1)\phi(y)\,\mathrm{e}^{\mathrm{i}\tilde{X}} - \delta\sigma\gamma\mathrm{i}A(x_1)\bar{v}_1(y)\,\mathrm{e}^{\mathrm{i}X}\cos Z + \text{c.c.} + \cdots,\\
w &= \delta\sigma A(x_1)\bar{w}_1(y)\,\mathrm{e}^{\mathrm{i}X}\sin Z + \text{c.c.} + \cdots,\\
p &= \epsilon\sigma B(x_1)\bar{p}_0(y)\,\mathrm{e}^{\mathrm{i}\tilde{X}} + \delta\sigma A(x_1)\bar{p}_1(y)\,\mathrm{e}^{\mathrm{i}X}\cos Z + \text{c.c.} + \cdots,
\end{aligned}\right\} \tag{2.8}$$

where ϵ and δ represent the magnitudes of the planar and oblique T-S waves, and $B(x_1)$ and $A(x_1)$ their (scaled) amplitude functions, respectively. Hereinafter c.c. represents the complex conjugate. For convenience, we have defined

$$\gamma = \big(\alpha_0^2 + \beta^2\big)^{1/2}.$$

The linear instability problem is described by a multi-layer structure consisting of five asymptotic regions: the potential-flow zone (i.e. 'upper-deck'), the main layer, the

Tollmien layer, the Stokes layer and the critical layer. These layers have thicknesses of orders σ^{-1}, 1, σ, σ^4 and σ^3, respectively (Bodonyi & Smith 1981). The solution in each of these regions can be obtained by expanding in terms of small parameter σ, and matching between different zones gives the leading-order dispersion relation

$$\lambda\tilde{c}_0 = \tilde{\alpha}_0, \quad \lambda c_0 = \gamma. \tag{2.9}$$

The above relations indicate that in order for $\tilde{c}_0 = c_0$, we require $\tilde{\alpha}_0 = \gamma$, which will be assumed to be the case hereinafter. The growth rates are found to be (see e.g. Goldstein & Durbin 1986; Wu *et al.* 1997)

$$2\tilde{\alpha}_0 c_0^{-1}\frac{B'}{B} = c_0(C^+ - C^-) + \lambda^2\tilde{\alpha}_0(2\tilde{\alpha}_0 c_0)^{-1/2} + \mathrm{i}\chi_b, \tag{2.10}$$

$$\gamma c_0^{-1}(\cos\theta + \sec\theta)\frac{A'}{A} = c_0(c^+ - c^-) + \lambda^2\gamma(2\alpha_0 c_0)^{-1/2} + \mathrm{i}\chi_a, \tag{2.11}$$

where we have put

$$\theta = \sin^{-1}\beta/\gamma, \quad Y_c = c_0/\lambda, \quad \mu = -c_0^2/4. \tag{2.12}$$

The first terms on the right-hand sides of (2.10) and (2.11) are the jumps across the critical layer, while the second terms are the viscous contributions from the Stokes layer adjacent to the wall. In the linear regime,

$$C^+ - C^- = 2\pi\tilde{\alpha}_0\mu Y_c, \quad c^+ - c^- = 2\pi\gamma\mu Y_c.$$

The real constants χ_a and χ_b in (2.10) and (2.11) represent $O(\sigma^3)$ corrections to the respective wavenumbers, and their specific values are not important as far as the development of the disturbance is concerned.

3. Nonlinear stage I

3.1. *The general case: imperfectly phase-locked modes*

Since the planar mode has a larger linear growth rate than the oblique ones, it is reasonable to assume that the former initially has a much larger magnitude than the latter, that is,

$$\delta \ll \epsilon \ll 1.$$

Similar to the Rayleigh waves considered in Wu & Stewart (1996) and many other critical-layer analyses of this type (Goldstein 1995*b*), the dominant nonlinear interaction takes place in the common critical layer, whilst the flow in the other layers remains linear to the required order so that the analysis of those layers is omitted. For the T-S waves of present interest, the critical layer is in equilibrium and viscosity dominated (Reid 1965; Bodonyi & Smith 1981). The balance between the inertia and viscous diffusion in the streamwise momentum equation shows that the transverse variable for this layer is

$$\eta = (y - y_c)/\sigma^3, \tag{3.1}$$

where $y_c = \sigma Y_c$.

The first nonlinear regime is reached when the jump across the critical layer produced by the phase-locked interaction appears at the same order as the linear jump in the oblique modes. An order-of-magnitude argument indicates that this occurs when

$$\epsilon = \sigma^{17/2}. \tag{3.2}$$

This determines the required threshold amplitude in order for the phase-locked interaction to exert a leading-order influence on the development of the oblique modes. We note that this key amplitude is asymptotically larger than the amplitude required for subharmonic resonance, i.e. $\epsilon = O(\sigma^{10})$ (cf. Mankbadi *et al.* 1993), which means that a subharmonic resonance could develop first if the correct modes are present. However, a difference of $O(\sigma^{-3/2}) \sim O(R^{3/20})$ is unlikely to be substantial in practical situations where R is typically about $O(10^3)$. Moreover to justify the relevance and utility of our theory, we refer to the comparison, albeit qualitative, with experiment at the end of § 7 (see also figure 7).

It is informative to present the arguments leading to scaling (3.2) as it highlights some interesting physical and mathematical features of the phase-locked interaction. The starting point is that both the streamwise and spanwise velocities of the oblique modes, u_{3D} and w_{3D} say, exhibit a simple-pole singularity at their critical layer (e.g. Mankbadi *et al.* 1993), so that their magnitude increases to $O(\delta\sigma^{-2})$ in the critical layer. Within this layer, the interaction between the planar and oblique modes produces a forcing term, $v_{2D}u_{3D,y}$, which is $O(\epsilon\delta\sigma^{-3})$, since the normal velocity of the planar mode $v_{2D} \sim \epsilon\sigma^2$. This forcing generates modes with difference and sum frequencies, both having an $O(\epsilon\delta\sigma^{-7})$ streamwise velocity, as can be deduced by balancing $v_{2D}u_{3D,y}$ with the inertia in the streamwise momentum equation. It turns out that the sum mode is confined within the critical layer, but the streamwise velocity of the difference mode, u_{d1} say, exhibits a jump across the critical layer. As a result, the difference mode spreads out to the Tollmien layer, where its streamwise velocity, u_d say, must be $O(\epsilon\delta\sigma^{-7})$, while the normal velocity is $O(\epsilon\delta\sigma^{-5})$ (as can be inferred from the continuity equation). It follows from the streamwise momentum equation that the pressure of the difference mode must have an $O(\epsilon\delta\sigma^{-6})$ magnitude. The crucial feature, which makes the present phase-locked interaction interesting and powerful, is that u_d exhibits a simple-pole singularity at the critical level so that the leading-order streamwise velocity of the difference mode, u_{d0} say, is $O(\epsilon\delta\sigma^{-9})$, much larger than the locally generated u_{d1}. The planar mode then interacts with this difference mode to produce the nonlinear forcing $v_{2D}u_{d0,y}$, which regenerates an oblique component with an $O(\delta\epsilon^2\sigma^{-14})$ streamwise velocity. This velocity produces a nonlinear jump across the critical layer. The phase-locked interaction starts to affect the evolution of the oblique modes when this jump becomes of the same order as the $O(\delta\sigma^3)$ linear jump. This balance leads to (3.2). The spanwise and normal velocities of this component are then found to be $O(\delta\sigma^3)$ and $O(\delta\sigma^7)$ from the spanwise momentum and continuity equations, respectively.

Guided by the argument above, we seek the expansion for the disturbance

$$\begin{aligned} u = {} & \epsilon\lambda B\, e^{i\tilde{X}} + \delta\sigma^{-2}\{U_1 + \sigma^5 U_2 + \cdots\} A\, e^{iX} \cos Z \\ & + \epsilon\delta\sigma^{-7}\{\sigma^{-2}U_{d0} + U_{d1} + \cdots\} A^* B E_d \cos Z + \text{c.c.} + \cdots, \end{aligned} \tag{3.3}$$

$$\begin{aligned} v = {} & \epsilon\{\sigma^2(-i\tilde{\alpha}_0 c_0)B + \cdots\}\, e^{i\tilde{X}} \\ & + \delta\big\{\sigma^2(1+\sigma a)(-i\gamma c_0) + \sigma^4 V_1 + \cdots + \sigma^7 V_2 + \cdots\big\} A\, e^{iX} \cos Z \\ & + \epsilon\delta\sigma^{-3}\{\sigma^{-2}V_{d0} + V_{d1} + \cdots\} A^* B E_d \cos Z + \cdots, \end{aligned} \tag{3.4}$$

$$\begin{aligned} w = {} & \delta\sigma^{-2}\{W_1 + \sigma^5 W_2 + \cdots\} A\, e^{iX} \sin Z \\ & + \delta\epsilon\sigma^{-7}\{\sigma^{-2}W_{d0} + W_{d1} + \cdots\} A^* B E_d \sin Z + \text{c.c.} + \cdots, \end{aligned} \tag{3.5}$$

$$p = \epsilon\sigma\tilde{\alpha}_0 B\, e^{i\tilde{X}} + \delta\sigma\gamma \cos\theta A\, e^{iX} \cos Z + \delta\epsilon\sigma^{-6} P_{d0} A^* B E_d \cos Z + \text{c.c.} + \cdots, \tag{3.6}$$

where a is an $O(1)$ constant (see Goldstein & Lee 1992; Mankbadi *et al.* 1993), the exact value of which is of no significance in our study. The terms proportional to

$$E_d = \exp\{i\sigma(\tilde{\alpha} - \alpha)(x - \sigma ct) - i\sigma^4 \tilde{\alpha} \Delta t\}$$

represent the difference mode, generated by the mutual interaction between the planar and oblique waves. Here we have omitted the sum mode and certain harmonic modes, as well as the two- and three-dimensional mean-flow distortions, since they do not contribute any nonlinear effect to the order considered in the present stage. Specifically, we note that while the mean flow due to the self-interaction of the planar mode is $O(\epsilon^2\sigma^{-2})$, which can be larger than the difference mode included in the expansion, the further interaction of an oblique mode with this mean flow generates a jump of $O(\delta\epsilon^2\sigma^{-7})$ in the streamwise velocity, which is much smaller than the $O(\delta\epsilon^2\sigma^{-14})$ jump produced by the difference mode.

The leading-order streamwise and spanwise velocities of the oblique T-S waves, U_1 and W_1, satisfy (Mankbadi *et al.* 1993; Wu 1993)

$$\left.\begin{aligned} \left[i\alpha_0\lambda\eta - \frac{\partial^2}{\partial\eta^2}\right] U_1 + \lambda(-i\gamma c_0) &= -i\alpha_0\gamma\cos\theta, \\ \left[i\alpha_0\lambda\eta - \frac{\partial^2}{\partial\eta^2}\right] W_1 &= \beta\gamma\cos\theta. \end{aligned}\right\} \tag{3.7}$$

These equations can be solved to obtain

$$U_1 = ic_0 \tan\theta \sin\theta\, \Pi^{(0)}, \quad W_1 = c_0 \sin\theta\, \Pi^{(0)}, \tag{3.8}$$

where we have defined

$$\Pi^{(n)} = \int_0^\infty \xi^n \exp(-i\eta\xi - \hat{s}\xi^3)\, d\xi, \quad \hat{s} = \tfrac{1}{3}(\lambda\alpha_0)^{-1}. \tag{3.9}$$

The leading-order velocity components of the difference mode, U_{d0} and W_{d0}, are driven by the pressure P_{d0}, as indicated by their governing equations (cf. Wu & Stewart 1996)

$$\left.\begin{aligned} \left[i\alpha_d\lambda(\eta - \eta_d) - \frac{\partial^2}{\partial\eta^2}\right] U_{d0} + \lambda V_{d0} &= -i\alpha_d P_{d0}, \\ \left[i\alpha_d\lambda(\eta - \eta_d) - \frac{\partial^2}{\partial\eta^2}\right] W_{d0} &= \beta P_{d0}, \end{aligned}\right\} \tag{3.10}$$

with

$$\alpha_d = \tilde{\alpha}_0 - \alpha_0, \quad \eta_d = \frac{\tilde{\alpha}_0 \Delta}{\alpha_d \lambda}. \tag{3.11}$$

V_{d0} and P_{d0} are the straightforward continuation of their corresponding solution in the Tollmien-layer, and have the form

$$V_{d0} = -i\gamma_d c_0 D, \quad P_{d0} = \gamma_d \cos\theta_d D, \tag{3.12}$$

where $\gamma_d = (\alpha_d^2 + \beta^2)^{1/2}$, and the constant D, which is a measure of the difference-mode amplitude, is to be found. Inserting (3.12) into (3.10), and solving the resulting equations, we find that

$$U_{d0} = ic_0 \tan\theta_d \sin\theta_d\, D\Pi_d^{(0)}, \quad W_{d0} = c_0 \sin\theta_d\, D\Pi_d^{(0)}, \tag{3.13}$$

where $\theta_d = \sin^{-1}\beta/\gamma_d$,

$$\Pi_d^{(n)} = \int_0^\infty \xi^n \exp(-\mathrm{i}(\eta-\eta_d)\xi - s_d\xi^3)\,\mathrm{d}\xi, \quad s_d = \tfrac{1}{3}(\lambda\alpha_d)^{-1}. \tag{3.14}$$

The second-order difference-mode terms, U_{d1}, V_{d1} and W_{d1}, are directly driven by the Reynolds stresses due to the interaction between the planar and oblique modes. They are governed by equations

$$\left.\begin{aligned} \mathrm{i}\alpha_d U_{d1} + \frac{\partial V_{d1}}{\partial\eta} + \beta W_{d1} = 0, \\ \left[\mathrm{i}\alpha_d\lambda(\eta-\eta_d) - \frac{\partial^2}{\partial\eta^2}\right]U_{d1} + \lambda V_{d1} = (\mathrm{i}\tilde{\alpha}_0 c_0)U^*_{1,\eta}, \\ \left[\mathrm{i}\alpha_d\lambda(\eta-\eta_d) - \frac{\partial^2}{\partial\eta^2}\right]W_{d1} = (\mathrm{i}\tilde{\alpha}_0 c_0)W^*_{1,\eta}. \end{aligned}\right\} \tag{3.15}$$

On eliminating U_{d1} and W_{d1}, we obtain

$$\left[\mathrm{i}\alpha_d\lambda(\eta-\eta_d) - \frac{\partial^2}{\partial\eta^2}\right]V_{d1,\eta\eta} = \mathrm{i}\tilde{\alpha}_0^2 c_0^2 \sin\theta\tan\theta\,\Pi^{*(2)}, \tag{3.16}$$

which has the solution

$$V_{d1,\eta\eta} = \mathrm{i}\frac{\tilde{\alpha}_0}{\alpha_d}c_0^3 \sin\theta\tan\theta \int_0^\infty\int_0^\infty \xi^2 \exp\left(-(\hat{s}+s_d)\xi^3 - s_d(\zeta-\xi)^3 + \mathrm{i}\eta_d\zeta - \mathrm{i}(\zeta-\xi)\eta\right)\mathrm{d}\xi\,\mathrm{d}\zeta. \tag{3.17}$$

There is a jump in $V_{d1,\eta}$, namely

$$J_d \equiv V_{d1,\eta}(\infty) - V_{d1,\eta}(-\infty) = 2\pi\mathrm{i}\frac{\tilde{\alpha}_0}{\alpha_d}c_0^3\sin\theta\tan\theta\, I_2, \tag{3.18}$$

where we have defined

$$I_n(\eta_d) = \int_0^\infty \xi^n \exp(-(\hat{s}+s_d)\xi^3 + \mathrm{i}\eta_d\xi)\,\mathrm{d}\xi. \tag{3.19}$$

Matching $V_{d1,\eta}$ with the Tollmien-layer solution determines

$$D = (-\mathrm{i}\gamma_d c_0)^{-1}\chi J_d; \tag{3.20}$$

the constant χ must be found by considering the solutions in the Tollmien layer, the main layer and the upper layer. This amounts to solving a boundary-value problem with inhomogeneous jump condition (cf. Wu & Stewart 1996). We omit the details (which are similar to those in Jennings 1997), and give only the final result

$$\chi = \frac{\gamma_d}{\lambda^2}\left(1 - \frac{\gamma_d}{\gamma}\right)^{-1}. \tag{3.21}$$

The amplitude of the difference mode becomes infinite if $\gamma_d = \gamma$ (i.e. if $\alpha_d = \alpha_0$). This is expected since it corresponds to subharmonic resonance, which appears as a singular limit of the phase-locked interaction. The relations (3.12) and (3.20) indicate that the pressure P_{d0} is driven by the nonlinear interaction within the critical layer through the velocity jump (3.18). Through P_{d0}, the effect of the interaction is 'transmitted' to a much lower order, forcing the large-amplitude velocity components U_{d0} and W_{d0}. It is this feature that makes the phase-locked interaction very effective (cf. Wu & Stewart 1996).

The planar mode interacts with the leading-order difference mode to regenerate the components at the fundamental frequency of the oblique modes, U_2, V_2 and W_2. Their orders of magnitude in the expansion (3.3)–(3.6) were determined by the scaling argument given above. It is found that (U_2, V_2, W_2) satisfies the equations (which are the steady version of those in Wu & Stewart 1996)

$$\left.\begin{aligned} &\mathrm{i}\alpha_0 U_2 + \frac{\partial V_2}{\partial \eta} + \beta W_2 = 0, \\ &\left[\mathrm{i}\alpha_0 \lambda \eta - \frac{\partial^2}{\partial \eta^2}\right] U_2 + \lambda V_2 = (\mathrm{i}\tilde{\alpha}_0 c_0)|B|^2 U^*_{d0,\eta}, \\ &\left[\mathrm{i}\alpha_0 \lambda \eta - \frac{\partial^2}{\partial \eta^2}\right] W_2 = (\mathrm{i}\tilde{\alpha}_0 c_0)|B|^2 W^*_{d0,\eta}, \end{aligned}\right\} \tag{3.22}$$

from which it follows that

$$\left[\mathrm{i}\alpha_0 \lambda \eta - \frac{\partial^2}{\partial \eta^2}\right] V_{2,\eta\eta} = 2\alpha_0 \gamma \mu c_0 + \mathrm{i}\tilde{\alpha}_0^2 c_0^2 \sin\theta_d \tan\theta_d D^* |B|^2 \Pi_d^{*(2)}. \tag{3.23}$$

Solving this, we find

$$V_{2,\eta\eta} = 2\gamma\mu c_0 \lambda^{-1} \Pi^{(0)} + \mathrm{i}\frac{\tilde{\alpha}_0}{\alpha_0} c_0^3 \sin\theta_d \tan\theta_d D^* |B|^2 \times \int_0^\infty \int_0^\infty \xi^2 \exp(-(\hat{s}+s_d)\xi^3 - \hat{s}(\zeta-\xi)^3 - \mathrm{i}\eta_d \xi - \mathrm{i}(\zeta-\xi)\eta)\, \mathrm{d}\xi\, \mathrm{d}\zeta. \tag{3.24}$$

Matching $V_{2,\eta}$ with the Tollmien-layer solution determines the jump

$$c^+ - c^- = V_{2,\eta}(\infty) - V_{2,\eta}(-\infty) = 2\pi\mu\gamma c_0 \lambda^{-1} + 2\pi\mathrm{i}\frac{\tilde{\alpha}_0}{\alpha_0} c_0^3 \sin\theta_d \tan\theta_d D^* |B|^2 I_2^*. \tag{3.25}$$

Substituting (3.25) into (2.11) yields the amplitude equation for the oblique modes

$$\frac{\mathrm{d}A}{\mathrm{d}x_1} = \kappa_a A + \mathrm{i}\Upsilon |B|^2 A, \tag{3.26}$$

where we have put

$$\kappa_a = c_0(\cos\theta + \sec\theta)^{-1}\left[\lambda^2(2\alpha_0 c_0)^{-1/2} + 2c_0^2\lambda^{-1}\mu\pi\right], \tag{3.27}$$

$$\Upsilon = -4\pi^2 c_0^9 (\alpha_0\alpha_d)^{-1} \sin\theta \tan\theta \sin\theta_d \tan\theta_d (\gamma - \gamma_d)^{-1} I_2^{*2} (\cos\theta + \sec\theta)^{-1}. \tag{3.28}$$

The planar wave still evolves linearly and its amplitude equation is

$$\frac{\mathrm{d}B}{\mathrm{d}x_1} = \kappa_b B, \tag{3.29}$$

so that

$$B = B_0 \exp(\kappa_b x_1) \quad \text{with} \quad \kappa_b = \tfrac{1}{2} c_0 [\lambda^2 (2\tilde{\alpha}_0 c_0)^{-1/2} + 2\pi c_0^2 \lambda^{-1} \mu].$$

Equation (3.26) is subjected to the initial condition

$$A \to A_0 \exp(\kappa_a x_1) \quad \text{as} \quad x_1 \to -\infty,$$

so that the solution matches to the linear stage upstream. It follows that

$$A = A_0 \exp\{\kappa_a x_1 + \mathrm{i}\Upsilon/(2\kappa_b)|B_0|^2 \exp(2\kappa_b x_1)\}. \tag{3.30}$$

In the general case, $\mathrm{i}\Upsilon$ is a complex number. Super-exponential growth arises if $Re(\mathrm{i}\Upsilon) > 0$. In figure 1, we plot the real part of $\mathrm{i}\Upsilon$ against the scaled phase-speed

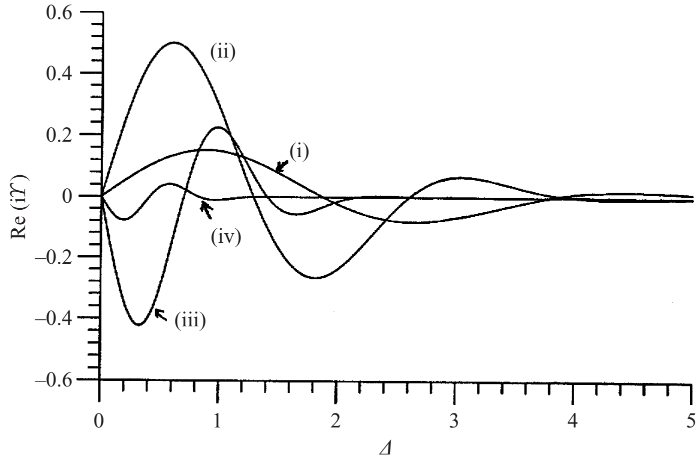

FIGURE 1. The real part of the coefficient $i\Upsilon$, $Re(i\Upsilon)$, *vs.* Δ for $\gamma = 0.12$: (i) $\theta = 80°$, (ii) $\theta = 70°$, (iii) $\theta = 50°$, (iv) $\theta = 35°$. The value $\gamma = 0.12$ is chosen to be within the main unstable region of the upper-branch scaling regime; see figure 1 of Wu *et al.* (1997).

mismatch, Δ, for a selection of obliqueness angle θ. It is clear that for each set of phase-locked modes, there is an 'optimal' value of Δ which gives the maximum $Re(i\Upsilon)$. On the other hand, $Re(i\Upsilon)$ decreases rapidly as Δ increases, indicating that a too large mismatch destroys the interaction under consideration. $Re(i\Upsilon)$ vanishes at an infinite number of values of Δ including $\Delta = 0$. Thus except at these special values, the amplitude A experiences super-exponential growth/decay. Thus similar to resonant-triad interaction (Goldstein 1994), the rapid development of the amplitude will eventually cause the non-equilibrium effect to become as important as viscosity within the critical layer. This occurs when the instantaneous growth rate of the oblique modes, $\sigma^4 A'/A \sim \sigma^4 \exp(2\kappa_b x_1)$, becomes of $O(\sigma^3)$, i.e. when

$$x_1 = O\left(\kappa_b^{-1} \log \sigma^{-1/2}\right). \tag{3.31}$$

This new stage will be investigated in § 4. Before that, let us consider the perfectly locked case $\Delta = 0$, as some interesting features will emerge.

3.2. *Perfectly phase-locked modes: wavelength-shortening induced nonlinearity*

When the planar and oblique modes are perfectly locked, $\Delta = 0$ and Υ is real. As a consequence, the nonlinear interaction affects only the phase angle of $A(x_1)$. Solution (3.30) indicates that the wavelength of the oblique modes 'shortens' exponentially, while its modulus continues to evolve exponentially as in the linear regime. At first sight, the above result seems to suggest that the phase-locked interaction is of no significance at all for the perfectly phase-locked modes. This however is not the case, because the rapid modulation of the phase induces velocity components, given by (3.44) and (3.45) below. These interact with the planar wave. The jump contributed by this interaction will appear at the same order as the linear jump when

$$x_1 = O\left(\kappa_b^{-1} \log \sigma^{-1/4}\right), \tag{3.32}$$

at which point, the phase modulation affects the development of the modulus. We thus introduce the shifted coordinate

$$\hat{x} = x_1 - \kappa_b^{-1} \log \sigma^{-1/4}. \tag{3.33}$$

For $\hat{x}=O(1)$, we may write $B(x_1)=\sigma^{-1/4}\hat{B}$, and the magnitude of the planar mode rises to

$$\hat{\epsilon}=\epsilon\sigma^{-1/4}=\sigma^{33/4}, \tag{3.34}$$

which is still asymptotically smaller than the required threshold magnitude of $O(\sigma^5)$ for a planar wave to become nonlinear (Goldstein & Durbin 1986). Thus $\hat{B}$ has the solution $\hat{B}=B_0\exp(\kappa_b\hat{x})$.

The disturbance in the main part of the flow can still be expanded as (2.8) except that ϵ is replaced by $\hat{\epsilon}$. Similar to the subharmonic resonance scenario considered by Wundrow *et al.* (1994), the amplitude function $A(\hat{x})$ of the oblique modes in this stage evolves over a shorter length scale than $\hat{B}$, and the solution for A takes the WKBJ form

$$A(\hat{x})=\hat{A}(\hat{x})\exp(\mathrm{i}\hat{\Theta}(\hat{x})/\sigma^{1/2}), \tag{3.35}$$

which is also implied by (3.30). Then we have the multiple-scale substitution

$$\frac{\partial}{\partial x}\rightarrow\sigma\alpha\frac{\partial}{\partial X}+\sigma\tilde{\alpha}\frac{\partial}{\partial\tilde{X}}+\sigma^{7/2}c^{-1}\hat{\Theta}'\frac{\partial}{\partial\hat{\Theta}}+\sigma^4c^{-1}\frac{\partial}{\partial\hat{x}}. \tag{3.36}$$

The overall flow structure remains the same as in stage I. Outside the critical layer, the flow is linear up to the order of our interest. A straightforward expansion in each of the layers and matching show that (cf. Wundrow *et al.* 1994; Wu *et al.* 1997)

$$\mathrm{i}\gamma c_0^{-1}(\cos\theta+\sec\theta)\frac{\mathrm{d}\hat{\Theta}}{\mathrm{d}\hat{x}}=c_0(\hat{a}^+-\hat{a}^-), \tag{3.37}$$

$$\gamma c_0^{-1}(\cos\theta+\sec\theta)\frac{\mathrm{d}\hat{A}}{\mathrm{d}\hat{x}}=c_0(\hat{c}^+-\hat{c}^-)+\lambda^2\gamma(2\alpha_0c_0)^{-1/2}A, \tag{3.38}$$

where the jumps $(\hat{a}^+-\hat{a}^-)$ and $(\hat{c}^+-\hat{c}^-)$ are to be determined by analysing the nonlinear dynamics within the critical layer. The expansion for the disturbance is

$$\begin{aligned}u=&\ \hat{\epsilon}\lambda\hat{B}\exp(\mathrm{i}\tilde{X})+\delta\sigma^{-2}\{\hat{U}_1+\sigma^{1/2}\hat{U}_2+\sigma^{1/2}\hat{U}_3+\sigma\hat{U}_4+\cdots\}\hat{A}\exp(\mathrm{i}\hat{\Theta}/\sigma^{1/2}+\mathrm{i}X)\cos Z\\&+\hat{\epsilon}\delta\sigma^{-7}\{\sigma^{-2}\hat{U}_{d0}+\sigma^{-3/2}\hat{U}_{d1}+\hat{U}_{d2}+\sigma^{1/2}\hat{U}_{d3}+\cdots\}\hat{A}^*\hat{B}\exp(-\mathrm{i}\hat{\Theta}/\sigma^{1/2})E_d\cos Z\\&+\mathrm{c.c.}+\cdots,\end{aligned} \tag{3.39}$$

$$\begin{aligned}v=&\ \hat{\epsilon}\sigma^2(-\mathrm{i}\tilde{\alpha}_0c_0)\hat{B}\exp(\mathrm{i}\tilde{X})+\delta\{\sigma^2(1+\sigma a)(-\mathrm{i}\gamma c_0)\\&+\sigma^4(-\mathrm{i}\gamma\lambda\eta)+\sigma^{13/2}\hat{V}_3+\sigma^7\hat{V}_4+\cdots\}\hat{A}\exp(-\mathrm{i}\hat{\Theta}/\sigma^{1/2}+\mathrm{i}X)\cos Z\\&+\hat{\epsilon}\delta\sigma^{-3}\{\sigma^{-2}\hat{V}_{d0}+\sigma^{-3/2}\hat{V}_{d1}+\hat{V}_{d2}+\sigma^{1/2}\hat{V}_{d3}+\cdots\}\hat{A}^*\hat{B}\exp(-\mathrm{i}\hat{\Theta}/\sigma^{1/2})E_d\cos Z\\&+\mathrm{c.c.}+\cdots,\end{aligned} \tag{3.40}$$

$$\begin{aligned}w=&\ \delta\sigma^{-2}\{\hat{W}_1+\sigma^{1/2}\hat{W}_2+\sigma^{9/2}\hat{W}_3+\sigma^5\hat{W}_4+\cdots\}\hat{A}\exp(\mathrm{i}\hat{\Theta}/\sigma^{1/2}+\mathrm{i}X)\sin Z\\&+\hat{\epsilon}\delta\sigma^{-7}\{\sigma^{-2}\hat{W}_{d0}+\sigma^{-3/2}\hat{W}_{d1}+\hat{W}_{d2}+\sigma^{1/2}\hat{W}_{d3}+\cdots\}\hat{A}^*\hat{B}\exp(-\mathrm{i}\hat{\Theta}/\sigma^{1/2})E_d\sin Z\\&+\mathrm{c.c.}+\cdots,\end{aligned} \tag{3.41}$$

$$\begin{aligned}p=&\ \hat{\epsilon}\sigma\tilde{\alpha}_0\hat{B}\exp(\mathrm{i}\tilde{X})+\delta\sigma\gamma\cos\theta\hat{A}\exp(\mathrm{i}\hat{\Theta}/\sigma^{1/2}+\mathrm{i}X)\cos Z\\&+\hat{\epsilon}\delta\sigma^{-6}\{\hat{P}_{d0}+\sigma^{1/2}\hat{P}_{d1}\}\hat{A}^*\hat{B}\exp(-\mathrm{i}\hat{\Theta}/\sigma^{1/2})E_d\cos Z+\mathrm{c.c.}+\cdots.\end{aligned} \tag{3.42}$$

The leading-order streamwise and spanwise velocities of the oblique modes, $\hat{U}_1$ and $\hat{W}_1$, are the same as given in (3.8), while for the difference mode,

$$\hat{P}_{d0}=P_{d0},\quad \hat{U}_{d0}=U_{d0},\quad \hat{W}_{d0}=W_{d0},\quad \hat{U}_{d2}=U_{d1},\quad \hat{W}_{d2}=W_{d1}.$$

The jump $(\hat{a}^+ - \hat{a}^-)$ is given by the second term in (3.25), and using it in (3.37), we obtain the equation for the phase $\hat{\Theta}$

$$\hat{\Theta}' = \Upsilon |\hat{B}|^2. \tag{3.43}$$

In order to derive the governing equation for the amplitude $\hat{A}$, we need to consider the second-order fundamental components in (3.39) and (3.41), $\hat{U}_2$ and $\hat{W}_2$, which satisfy the equations

$$\left[\mathrm{i}\alpha_0 \lambda \eta - \frac{\partial^2}{\partial \eta^2} \right] \hat{U}_2 = c_0 \tan\theta \sin\theta \, \hat{\Theta}'(\hat{x}) \Pi^{(0)}, \tag{3.44}$$

$$\left[\mathrm{i}\alpha_0 \lambda \eta - \frac{\partial^2}{\partial \eta^2} \right] \hat{W}_2 = -\mathrm{i} c_0 \sin\theta \, \hat{\Theta}'(\hat{x}) \Pi^{(0)}. \tag{3.45}$$

The solution is found to be

$$\hat{U}_2 = (\alpha_0 \lambda)^{-1} c_0 \tan\theta \sin\theta \, \hat{\Theta}'(\hat{x}) \, \Pi^{(1)}, \quad \hat{W}_2 = -\mathrm{i}(\alpha_0 \lambda)^{-1} c_0 \sin\theta \, \hat{\Theta}'(\hat{x}) \, \Pi^{(1)}. \tag{3.46}$$

These velocity components are induced by the modulation of the phase $\hat{\Theta}(\hat{x})$, since their forcing terms are proportional to $\hat{\Theta}'$. A similar scenario arises in the cases of oblique-mode (Wu *et al.* 1997), and resonant-triad interactions (Wundrow *et al.* 1994).

As with V_{d0} and P_{d0}, the solution for $\hat{V}_{d1}$ and $\hat{P}_{d1}$ can be written as

$$\hat{V}_{d1} = (-\mathrm{i}\gamma_d c_0 \hat{D}) \hat{\Theta}', \quad \hat{P}_{d1} = \gamma_d \cos\theta_d \hat{D} \hat{\Theta}', \tag{3.47}$$

with constant $\hat{D}$ to be determined later. The governing equations for $\hat{U}_{d1}$, $\hat{V}_{d1}$ and $\hat{W}_{d1}$ are

$$\left[\mathrm{i}\alpha_d \lambda \eta - \frac{\partial^2}{\partial \eta^2} \right] \hat{U}_{d1} + \lambda \hat{V}_{d1} = -\mathrm{i}\alpha_d \hat{P}_{d1} + \mathrm{i}\hat{\Theta}' \hat{U}_{d0}, \tag{3.48}$$

$$\left[\mathrm{i}\alpha_d \lambda \eta - \frac{\partial^2}{\partial \eta^2} \right] \hat{W}_{d1} = \beta \hat{P}_{d1} + \mathrm{i}\hat{\Theta}' \hat{W}_{d0}, \tag{3.49}$$

which we solve to obtain

$$\hat{U}_{d1} = \mathrm{i} c_0 \sin\theta_d \tan\theta_d \left\{ \hat{D} \Pi_d^{(0)} + \mathrm{i}(\alpha_d \lambda)^{-1} D \Pi_d^{(1)} \right\} \hat{\Theta}', \tag{3.50}$$

where D is given by (3.20).

The back effect of the phase modulation on the modulus is facilitated by $\hat{U}_2$ and $\hat{W}_2$ (the wave-length shortening induced velocities) interacting with the planar wave to generate those velocity components of the difference mode that are represented by $\hat{U}_{d3}$, $\hat{V}_{d3}$ and $\hat{W}_{d3}$ in (3.39)–(3.41). They satisfy

$$\left.\begin{aligned} &\mathrm{i}\alpha_d \hat{U}_{d3} + V_{d3,\eta} + \beta \hat{W}_{d3} = 0. \\ &\left[\mathrm{i}\alpha_d \lambda \eta - \frac{\partial^2}{\partial \eta^2} \right] \hat{U}_{d3} + \lambda \hat{V}_{d3} = (\mathrm{i}\tilde{\alpha}_0 c_0) \hat{U}^*_{2,\eta} + \mathrm{i}\hat{\Theta}' \hat{U}_{d2}, \\ &\left[\mathrm{i}\alpha_d \lambda \eta - \frac{\partial^2}{\partial \eta^2} \right] \hat{W}_{d3} = (\mathrm{i}\tilde{\alpha}_0 c_0) \hat{W}^*_{2,\eta} + \mathrm{i}\hat{\Theta}' \hat{W}_{d2}. \end{aligned}\right\} \tag{3.51}$$

It follows that

$$\left[\mathrm{i}\alpha_d \lambda \eta - \frac{\partial^2}{\partial \eta^2} \right] \hat{V}_{d3,\eta\eta} = -\frac{\tilde{\alpha}_0}{\alpha_0} c_0^3 \sin\theta \tan\theta \hat{\Theta}' \Pi^{*(3)} + R_s, \tag{3.52}$$

where

$$R_s = -\frac{\tilde{\alpha}_0}{\alpha_d} c_0^3 \sin\theta \tan\theta\, \hat{\Theta}' \int_0^\infty \int_0^\infty \xi^2 \exp(-(\hat{s}+s_d)\xi^3 - s_d(\zeta-\xi)^3 - \mathrm{i}(\zeta-\xi)\eta)\, \mathrm{d}\xi\, \mathrm{d}\zeta.$$

Solving (3.52), we find that

$$I_d \equiv \hat{V}_{d3,\eta}(\infty) - \hat{V}_{d3,\eta}(-\infty) = -2\pi \frac{\tilde{\alpha}_0}{\alpha_0 \alpha_d^2} c_0^4 \sin\theta \tan\theta\, \hat{\Theta}' I_3(0). \tag{3.53}$$

This jump induces the pressure $\hat{P}_{d1}$, which in turn drives $\hat{U}_{d1}$ and $\hat{W}_{d1}$. Matching $\hat{V}_{d3,\eta}$ to the solution in the Tollmien layer shows that

$$\hat{D} = (-\mathrm{i}\gamma_d c_0)^{-1} \chi I_d$$

with χ being given by (3.21).

Next, we solve for $\hat{V}_4$ in (3.40), which is required to determine the jump $(\hat{c}^+ - \hat{c}^-)$. It is found that the governing equation for $\hat{V}_4$ is

$$\left[\mathrm{i}\alpha_0 \lambda \eta - \frac{\partial^2}{\partial \eta^2}\right] \hat{V}_{4,\eta\eta} = 2\alpha_0 c_0 \gamma \mu - \mathrm{i}\hat{\Theta}' \hat{V}_{3,\eta\eta} + (\gamma^2 c_0)|\hat{B}|^2 \hat{U}^*_{d1,\eta\eta}, \tag{3.54}$$

where $\hat{V}_{3,\eta\eta} = V_{2,\eta\eta}$ as given by (3.24). Solving (3.54) by Fourier transform, we obtain

$$\hat{c}^+ - \hat{c}^- \equiv \hat{A}\big[\hat{V}_{4,\eta}(+\infty) - \hat{V}_{4,\eta}(-\infty)\big] = 2\gamma c_0 \lambda^{-1} \mu \pi \hat{A} + q\hat{\Theta}'|\hat{B}|^2 \hat{A}, \tag{3.55}$$

where

$$q = -8\pi^2 (\alpha \alpha_d)^{-2} c_0^8 \sin\theta \tan\theta \sin\theta_d \tan\theta_d I_2(0) I_3(0) \left(1 - \frac{\gamma_d}{\gamma}\right)^{-1}. \tag{3.56}$$

Substituting (3.55) into (3.38), we obtain the evolution equation for the modulus $\hat{A}$:

$$\hat{A}' = \kappa_a \hat{A} + \hat{\Upsilon}_a \hat{\Theta}' |\hat{B}|^2 \hat{A}, \tag{3.57}$$

with $\hat{\Upsilon}_a = \gamma^{-1} c_0^2\, q (\cos\theta + \sec\theta)^{-1}$. Equation (3.57) is coupled to (3.43), indicating an interplay between the phase and modulus. The nonlinear term in (3.57) can be interpreted as arising from wavelength ‘shortening’ or dilation. Inserting (3.43) into (3.57) yields a single equation governing the modulus

$$\hat{A}' = \kappa_a \hat{A} + \hat{\Upsilon} |\hat{B}|^4 \hat{A}, \tag{3.58}$$

where

$$\hat{\Upsilon} \equiv \Upsilon \hat{\Upsilon}_a = 32\pi^4 c_0^6 \sin^2\theta \tan^2\theta \sin^2\theta_d \tan^2\theta_d I_3(0) (\cos\theta + \sec\theta)^{-2} (\gamma - \gamma_d)^{-2}. \tag{3.59}$$

The solution for $\hat{A}$ is

$$\hat{A} = A_0 \exp\{\kappa_a \hat{x} + \hat{\Upsilon}/(4\kappa_b)|B_0|^4 \exp(4\kappa_b \hat{x})\}. \tag{3.60}$$

Now since the coefficient $\hat{\Upsilon}$ is always positive, the amplitude undergoes super-exponential growth, regardless of the obliqueness angle θ. Note that the super-exponential growth, once started, is much faster than the non-zero mismatch case, since the rate is proportional to $\exp(4\kappa_b \hat{x})$ as opposed to $\exp(2\kappa_b \hat{x})$. The non-equilibrium effect associated with this rapid amplification becomes a leading-order effect when

$$\hat{x} = O\big(\kappa_b^{-1} \log \sigma^{-1/4}\big),$$

which from (3.33) is equivalent to (3.31), namely,

$$x_1 = O\big(\kappa_b^{-1} \log \sigma^{-1/2}\big).$$

Thus, the disturbance of perfectly phase-locked modes enters the non-equilibrium stage at the same downstream location as in the imperfectly phase-locked case.

A similar analysis can be performed for other values of Δ at which $Re(\mathrm{i}\Upsilon)=0$ to show that the disturbance follows the same evolution route as for $\Delta=0$.

4. Nonlinear stage II: WKBJ regime with a non-equilibrium critical layer

We have shown that with or without phase-speed mismatching, the disturbance evolves, albeit via somewhat different routes, into the non-equilibrium critical-layer regime when

$$x_1 = O\left(\kappa_b^{-1}\log\sigma^{-1/2}\right).$$

Introduce now

$$x^\dagger = x_1 - \kappa_b^{-1}\log\sigma^{-1/2} = \hat{x} - \kappa_b^{-1}\log\sigma^{-1/4}. \tag{4.1}$$

In this regime, the magnitude of the planar mode rises to

$$\bar{\epsilon} = \sigma^8,$$

while the magnitude of the oblique modes becomes of order $\bar{\delta} \equiv \delta\,\mathrm{e}^{\Upsilon/\sigma}$ if $\Delta \neq 0$ or $\delta\,\mathrm{e}^{\hat{\Upsilon}/\sigma}$ if $\Delta=0$. We shall assume that $\bar{\delta}$ is sufficiently small such that the back effect of the oblique modes on the planar mode remains negligible. Then the latter continues to grow exponentially with its amplitude function $B^\dagger = B_0\exp(\kappa_b x^\dagger)$. The amplitude function of the oblique modes now takes the WKBJ form

$$A = \bar{A}(x^\dagger)\exp(\Phi(x^\dagger)/\sigma), \tag{4.2}$$

where Φ is a complex function. The development of the oblique modes is primarily characterized by $\Phi(x^\dagger)$ with $\bar{A}(x^\dagger)$ being of secondary importance. Based on the analogy of the present form of solution with that in § 3, we can infer that Φ is related to the jump across the critical layer via

$$\gamma c_0^{-1}(\cos\theta + \sec\theta)\frac{\mathrm{d}\Phi}{\mathrm{d}x^\dagger} = c_0(\bar{a}^+ - \bar{a}^-), \tag{4.3}$$

which is obtained by replacing $\mathrm{i}\hat{\Theta}$ in (3.37) by Φ. The expansion in the critical layer still takes the form (3.3)–(3.6) except that ϵ and $\hat{\delta}$ are replaced by $\bar{\epsilon}$ and $\hat{\delta}$, respectively. The leading-order terms, U_1 and W_1, now satisfy

$$\left.\begin{aligned} \left[\mathrm{i}\alpha_0\lambda\eta + \Phi' - \frac{\partial^2}{\partial\eta^2}\right]U_1 + \lambda(-\mathrm{i}\gamma c_0) &= -\mathrm{i}\alpha_0\gamma\cos\theta, \\ \left[\mathrm{i}\alpha_0\lambda\eta + \Phi' - \frac{\partial^2}{\partial\eta^2}\right]W_1 &= \beta\gamma\cos\theta. \end{aligned}\right\} \tag{4.4}$$

The non-equilibrium effect is reflected by the additional term Φ' in the operator. The above equations have the solution

$$U_1 = \mathrm{i}c_0\tan\theta\sin\theta\,\bar{\Pi}^{(0)}, \quad W_1 = c_0\sin\theta\,\bar{\Pi}^{(0)}, \tag{4.5}$$

where we have defined

$$\bar{\Pi}^{(n)} = \int_0^\infty \xi^n\exp(-\mathrm{i}\eta\xi - (\alpha_0\lambda)^{-1}\Phi'\xi - \hat{s}\xi^3)\,\mathrm{d}\xi. \tag{4.6}$$

The solution indicates that owing to the non-equilibrium effect, the normal distribution of the leading-order streamwise and spanwise velocities in the present stage undergoes deformation as their amplitudes evolve, i.e. both the amplitude and the 'shape' of the oblique modes are affected by the phase-locked interaction. This is in contrast to stage I, in which the interaction affects the amplitude only, while the 'shape' of the disturbance is still determined by linear dynamics.

The governing equations for U_{d0} and W_{d0} are

$$\left.\begin{aligned} \left[\mathrm{i}\alpha_d\lambda(\eta-\eta_d)+\Phi'^*-\frac{\partial^2}{\partial\eta^2}\right]U_{d0}+\lambda V_{d0}&=-\mathrm{i}\alpha_d P_{d0},\\ \left[\mathrm{i}\alpha_d\lambda(\eta-\eta_d)+\Phi'^*-\frac{\partial^2}{\partial\eta^2}\right]W_{d0}&=\beta P_{d0}. \end{aligned}\right\} \tag{4.7}$$

Similar to the previous stage, V_{d0} and P_{d0} are given by

$$V_{d0}=(-\lambda\gamma_d c_0 D^\dagger),\quad P_{d0}=\gamma_d\cos\theta_d D^\dagger, \tag{4.8}$$

where

$$D^\dagger=(-\mathrm{i}\gamma_d c_0)^{-1}\chi J_d^\dagger, \tag{4.9}$$

with $J_d^\dagger$ given by (4.14). It is easy to show that

$$U_{d0}=\mathrm{i}c_0\tan\theta_d\sin\theta_d\, D^\dagger\bar{\Pi}_d^{(0)},\quad W_{d0}=c_0\sin\theta_d\, D^\dagger\bar{\Pi}_d^{(0)}, \tag{4.10}$$

with

$$\bar{\Pi}_d^{(n)}=\int_0^\infty \xi^n\exp(-\mathrm{i}(\eta-\eta_d)\xi-(\alpha_d\lambda)^{-1}\Phi'^*\xi-s_d\xi^3)\,\mathrm{d}\xi. \tag{4.11}$$

The second-order velocity components of the difference mode, U_{d1}, V_{d1} and W_{d1}, are governed by equations

$$\left.\begin{aligned} \mathrm{i}\alpha_d U_{d1}+\frac{\partial V_{d1}}{\partial\eta}+\beta W_{d1}&=0,\\ \left[\mathrm{i}\alpha_d\lambda(\eta-\eta_d)+\Phi'^*-\frac{\partial^2}{\partial\eta^2}\right]U_{d1}+\lambda V_{d1}&=(\mathrm{i}\tilde{\alpha}_0c_0)U^*_{1,\eta\eta},\\ \left[\mathrm{i}\alpha_d\lambda(\eta-\eta_d)+\Phi'^*-\frac{\partial^2}{\partial\eta^2}\right]W_{d1}&=(\mathrm{i}\tilde{\alpha}_0c_0)W^*_{1,\eta\eta}, \end{aligned}\right\} \tag{4.12}$$

which, after eliminating U_{d1} and W_{d1}, give

$$\left[\mathrm{i}\alpha_d\lambda(\eta-\eta_d)+\Phi'^*-\frac{\partial^2}{\partial\eta^2}\right]V_{d1,\eta\eta}=\mathrm{i}\tilde{\alpha}_0^2c_0^2\sin\theta\tan\theta\,\bar{\Pi}^{*(2)}. \tag{4.13}$$

After solving this equation, it is found that there is a jump in $V_{d1,\eta}$, namely,

$$\begin{aligned} J_d^\dagger&\equiv V_{d1,\eta}(\infty)-V_{d1,\eta}(-\infty)\\ &=2\pi\mathrm{i}\frac{\tilde{\alpha}_0}{\alpha_d}c_0^3\sin\theta\tan\theta\int_0^\infty\xi^2\exp(-(\hat{s}+s_d)\xi^3-(\alpha_d^{-1}+\alpha_0^{-1})\lambda^{-1}\Phi'^*\xi+\mathrm{i}\eta_d\xi)\,\mathrm{d}\xi. \end{aligned} \tag{4.14}$$

The jump $(\bar{a}^+-\bar{a}^-)$ (see (4.3)) is obtained by considering $V_{2,\eta\eta}$, which satisfies

$$\left[\mathrm{i}\alpha_0\lambda\eta+\Phi'-\frac{\partial^2}{\partial\eta^2}\right]V_{2,\eta\eta}=\mathrm{i}\tilde{\alpha}_0^2c_0^2\sin\theta_d\tan\theta_d D^{\dagger *}|B^\dagger|^2\bar{\Pi}_d^{*(2)}. \tag{4.15}$$

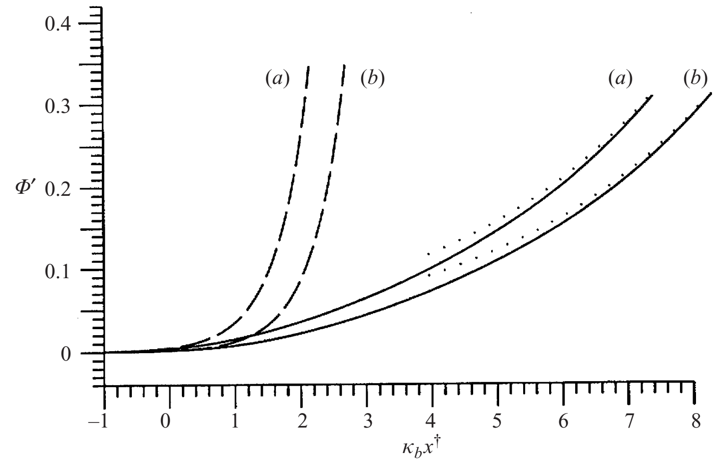

FIGURE 2. Solid lines, the super-exponential growth rate Φ' *vs.* $\kappa_b x^\dagger$ (*a*) $\theta = 40°$, (*b*) $\theta = 70°$. Dashed lines, the far upstream behaviour (4.18). Dotted lines, the far downstream behaviour (4.19).

Solving this and matching $V_{2,\eta}$ to the Tollmien-layer solution, we obtain

$$\bar{a}^+ - \bar{a}^- = 2\mathrm{i}\frac{\tilde{\alpha}_0}{\alpha_0} c_0^3 \sin\theta_d \tan\theta_d D^{\dagger *} |B^\dagger|^2 \times \int_0^\infty \xi^2 \exp(-(\hat{s} + s_d)\xi^3 - (\alpha_d^{-1} + \alpha_0^{-1})\lambda^{-1}\Phi'\xi - \mathrm{i}\eta_d\xi)\,\mathrm{d}\xi. \quad (4.16)$$

Inserting the above result into (4.3), we arrive at a transcendental equation for (the local growth rate) Φ',

$$\Phi' = \frac{\mathrm{i}\Upsilon}{I_2^{*2}} \left\{ \int_0^\infty \xi^2 \exp(-(\hat{s} + s_d)\xi^3 - \left(\alpha_d^{-1} + \alpha_0^{-1}\right)\lambda^{-1}\Phi'\xi - \mathrm{i}\eta_d\xi)\,\mathrm{d}\xi \right\}^2 |B^\dagger|^2. \quad (4.17)$$

Since $B^\dagger = B_0\,\mathrm{e}^{\kappa_b x^\dagger}$, it follows that

$$\Phi' \to \mathrm{i}\Upsilon |B_0|^2 \exp(2\kappa_b x^\dagger) + \hat{\Upsilon} |B_0|^4 \exp(4\kappa_b x^\dagger) \quad \text{as} \quad x^\dagger \to -\infty, \quad (4.18)$$

therefore matching to the solution in the previous stage (cf. (3.30) and (3.60)). On the other hand, as $x^\dagger \to \infty$,

$$\Phi' \to \Phi_\infty \exp\left(\tfrac{2}{7}\kappa_b x^\dagger\right) \quad \text{with} \quad \Phi_\infty = (\mathrm{i}\Upsilon / I_2^{*2})^{1/7} (\alpha_d \alpha_0 \lambda / \tilde{\alpha}_0)^{6/7}. \quad (4.19)$$

For $x^\dagger = O(1)$, equation (4.17) is solved using the Newton-Raphson method and representative results are shown in figure 2 for $\gamma = 0.12$, $\theta = 40^o$ and $\theta = 70^o$ with the phase-speed mismatch parameter Δ being the optimal value for each θ. The non-equilibrium effect inhibits the amplification, converting the growth rate from being $\exp(2\kappa_b x^\dagger)$ in the equilibrium regime to $\exp(2\kappa_b x^\dagger/7)$. However, it should be noted that the latter behaviour may not be attainable if the initial magnitude of the oblique modes is not sufficiently small. In that case, the disturbance enters the fully interactive stage before the asymptote (4.19) is reached (see below). In Figure 3, we plot the normal distribution of the leading-order streamwise velocity of the oblique modes. The deformation of the modal shape is characterized by 'thickening' of the region in which the disturbance concentrates, but there is little shift of the position of the velocity maximum.

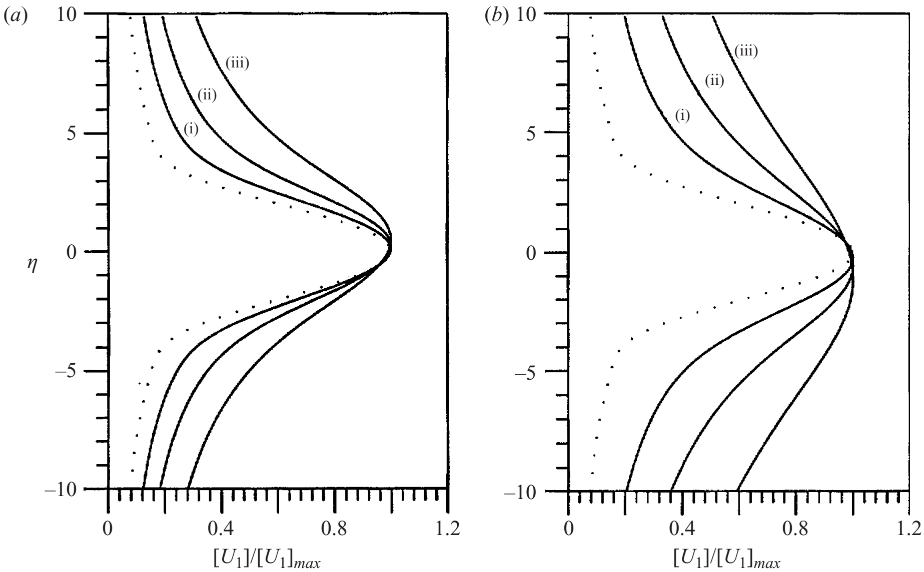

FIGURE 3. Deformation of the modal shape in the WKBJ regime illustrated by the modulus of the streamwise velocity, $|U_1|$, at different downstream locations. (*a*) $\theta = 40°$, $\kappa_b x^\dagger = 4$ (i), 6 (ii), 8 (iii). (*b*) $\theta = 70°$, $\kappa_b x^\dagger = 4$ (i), 6 (ii), 8 (iii). The dotted line represents the distribution in the linear regime.

5. Nonlinear stage III: the fully interactive regime

5.1. *Scaling and amplitude equations*

The continued growth of the oblique modes will eventually lead to a new stage in which the self-interaction between the oblique modes becomes as important as the phase-locked interaction considered earlier. This occurs when the unscaled magnitude of the oblique modes reaches $O(\sigma^7)$, i.e. (cf. Wu *et al.* 1997)

$$\tilde{\delta} \equiv \bar{\delta} \exp(\Phi(x_s^\dagger)/\sigma) = \sigma^7.$$

The above relation indicates that the initial magnitude of the oblique modes determines the downstream location $x_s^\dagger$ near which the fully interactive regime commences. The development is described by (4.17) as long as $(x_s^\dagger - x^\dagger) \gg O(\sigma)$. However, in the $O(\sigma)$ neighbourhood of $x_s^\dagger$, the WKBJ solution ceases to be valid, because the oblique modes produce a back reaction on the planar wave so that the latter no longer evolves linearly. Both the planar and oblique modes evolve over the same fast scale

$$\tilde{x} = \left(x^\dagger - x_s^\dagger\right)/\sigma. \tag{5.1}$$

The magnitude of the planar mode remains $O(\sigma^8)$, namely

$$\tilde{\epsilon} = \sigma^8. \tag{5.2}$$

The solution in the main part of the boundary layer again takes the form (2.8) provided that ϵ and δ are replaced by $\tilde{\epsilon}$ and $\tilde{\delta}$, respectively, and $A(x_1)$ and $B(x_1)$ by $\tilde{A}(\tilde{x})$ and $\tilde{B}(\tilde{x})$, respectively. The flow outside the critical layer remains linear and inviscid to the order of our interest, and the solution can be sought by a straightforward expansion. Matching the solutions in different regions gives (see Wu

1995)

$$\mathrm{i}c_0^{-1}(\cos\theta+\sec\theta)\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tilde{x}}=\gamma^2 c_0(\tilde{c}^+-\tilde{c}^-), \tag{5.3}$$

$$2\mathrm{i}c_0^{-1}\frac{\mathrm{d}\tilde{B}}{\mathrm{d}\tilde{x}}=\tilde{\alpha}_0^2 c_0(\tilde{C}^+-\tilde{C}^-), \tag{5.4}$$

where the jumps $(\tilde{c}^+-\tilde{c}^-)$ and $(\tilde{C}^+-\tilde{C}^-)$ are to be determined by considering the nonlinear interactions in the critical layer.

To illustrate the main feature of this regime and also to facilitate the matching with the next stage, we write the first few terms in the expansion within the critical layer:

$$u=\tilde{\epsilon}\lambda\tilde{B}\,\mathrm{e}^{\mathrm{i}\tilde{X}}+\tilde{\delta}\sigma^{-2}\{(1+\sigma a)\tilde{U}_1+\sigma^2(\lambda\sec\theta\tilde{A})+\cdots\}\,\mathrm{e}^{\mathrm{i}X}\cos Z+\text{c.c.}+\cdots, \tag{5.5}$$

$$v=\tilde{\epsilon}\sigma^2(-\mathrm{i}\tilde{\alpha}_0 c_0)\tilde{B}\,\mathrm{e}^{\mathrm{i}\tilde{X}} +\tilde{\delta}\sigma^2\{(1+\sigma a)(-\mathrm{i}\gamma c_0)\tilde{A}+\sigma^2(-\mathrm{i}\gamma\lambda\tilde{A})\,\eta+\cdots\}\,\mathrm{e}^{\mathrm{i}X}\cos Z+\text{c.c.}+\cdots, \tag{5.6}$$

$$w=\tilde{\delta}\sigma^{-2}\{\tilde{W}_1+\cdots\}\,\mathrm{e}^{\mathrm{i}X}\sin Z+\text{c.c.}+\cdots, \tag{5.7}$$

$$p=\tilde{\epsilon}\sigma\tilde{B}\,\mathrm{e}^{\mathrm{i}\tilde{X}}+\tilde{\delta}\sigma\gamma\cos\theta\tilde{A}\,\mathrm{e}^{\mathrm{i}X}\cos Z+\text{c.c.}+\cdots. \tag{5.8}$$

The leading-order streamwise and spanwise velocities, $\tilde{U}_1$ and $\tilde{W}_1$, are governed by equations

$$\left[\frac{\partial}{\partial\tilde{x}}+\mathrm{i}\alpha_0\lambda\eta-\frac{\partial^2}{\partial\eta^2}\right]\tilde{U}_1+\lambda(-\mathrm{i}\gamma c_0\tilde{A})=-\mathrm{i}\alpha_0\gamma\cos\theta\tilde{A},$$

$$\left[\frac{\partial}{\partial\tilde{x}}+\mathrm{i}\alpha_0\lambda\eta-\frac{\partial^2}{\partial\eta^2}\right]\tilde{W}_1=\beta\gamma\cos\theta\tilde{A}.$$

The non-equilibrium effect is now represented by $\partial/\partial\tilde{x}$ in the operator. The above equations have the solution

$$\tilde{U}_1=\mathrm{i}\gamma^2\sin^2\theta\,\tilde{\Pi},\quad \tilde{W}_1=\gamma^2\sin\theta\cos\theta\,\tilde{\Pi}, \tag{5.9}$$

with

$$\tilde{\Pi}=\int_0^\infty\tilde{A}(\tilde{x}-\xi)\exp(-\mathrm{i}\alpha_0\lambda\eta\xi-s\xi^3)\,\mathrm{d}\xi,\quad s=\tfrac{1}{3}\alpha_0^2\lambda^2. \tag{5.10}$$

The nonlinear interactions within the critical layer are the same as in Wu & Stewart (1996) and Wu (1996) so that the jumps across the critical layer, (c^+-c^-) and (C^+-C^-), can be borrowed from there without performing a detailed analysis. After substituting them into (5.3) and (5.4), we obtain the amplitude equations

$$\frac{\mathrm{d}\tilde{A}}{\mathrm{d}\tilde{x}}=\mathrm{i}\tilde{\Upsilon}_p\int_0^\infty\int_0^\infty K_p(\xi,\eta|s))\exp(-\mathrm{i}\alpha_0\Delta(\xi+\eta))\tilde{B}(\tilde{x}-\hat{\sigma}_d\xi)\tilde{B}^*(\tilde{x}-\xi-\hat{\sigma}\eta)\tilde{A}(\tilde{x}-\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+\mathrm{i}\tilde{\Upsilon}_a\int_0^\infty\int_0^\infty K_a(\xi,\eta|s)\tilde{A}(\tilde{x}-\xi)\tilde{A}(\tilde{x}-\xi-\eta)\tilde{A}^*(\tilde{x}-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{5.11}$$

$$\frac{\mathrm{d}\tilde{B}}{\mathrm{d}\tilde{x}}=\mathrm{i}\tilde{\Upsilon}_b\int_0^\infty\int_0^\infty K_b(\xi,\eta|s)\exp(-\mathrm{i}\tilde{\alpha}_0\Delta(\xi+\eta))\tilde{A}(\tilde{x}-\xi)\tilde{B}(\tilde{x}-\xi-\eta)\tilde{A}^*(\tilde{x}-\nu_s\xi-\nu_0\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$+\mathrm{i}\tilde{\Upsilon}_c\int_0^\infty\int_0^\infty K_c(\xi,\eta|s)\exp(-\mathrm{i}\tilde{\alpha}_0\Delta\xi)\tilde{B}(\tilde{x}-\xi)\tilde{A}(\tilde{x}-\xi-\eta)\tilde{A}^*(\tilde{x}-\nu_s\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{5.12}$$

where

$$K_p(\xi,\eta|\Lambda) = \xi^2\eta^2 \exp\{-\Lambda\hat{\sigma}_d^2(\xi^3+\eta^3)\}, \tag{5.13}$$

$$\tilde{\Upsilon}_p = -4\pi^2\alpha_d^4\alpha_0^4\gamma\gamma_d c_0^3 \sin^2\theta \sin^2\theta_d(\gamma-\gamma_d)^{-1}(\cos\theta+\sec\theta)^{-1}, \tag{5.14}$$

$$\tilde{\Upsilon}_a = -\pi\gamma^2\alpha_0^3\lambda^3 c_0^5 \sin^2\theta(\cos\theta+\sec\theta)^{-1}, \tag{5.15}$$

$$\tilde{\Upsilon}_b = \tilde{\Upsilon}_c = -\tfrac{1}{2}\pi\alpha_0^3\lambda^3 c_0^5\gamma^2 \sin^2\theta, \tag{5.16}$$

$$\left.\begin{aligned} \hat{\sigma} &= \alpha_0/\tilde{\alpha}_0, \quad \hat{\sigma}_d = \alpha_d/\tilde{\alpha}_0, \quad \hat{\sigma}_s = 1+\hat{\sigma}_0, \\ \nu_0 &= \tilde{\alpha}_0/\alpha_0, \quad \nu_d = \alpha_d/\alpha_0, \quad \nu_s = 1+\nu_0. \end{aligned}\right\} \tag{5.17}$$

The kernel $K_a(\xi,\eta|s)$ is given by (3.85) of Wu *et al.* (1993) while K_b and K_c are given by (A.1) and (A.2) of Wu & Stewart (1996) (with σ, σ_d etc. being replaced by $\hat{\sigma}$, $\hat{\sigma}_d$ as defined in (5.17)). Here the terms representing the linear growth do not enter the amplitude equations since the disturbance now evolves over a much shorter streamwise scale. It is worth comparing the coupled amplitude equations (5.11)–(5.12) with those for subharmonic resonance (Goldstein & Lee 1992; Goldstein 1995*a*; Wu 1995). The main difference is that the cubic term with kernel K_p now replaces a quadratic term to provide the vital catalytic effect of promoting the growth of the oblique modes. The term with kernel K_a, which represents the mutual interaction of the oblique modes (Goldstein & Choi 1989; Wu *et al.* 1993, 1997), remains the same. The terms describing the back action of the oblique modes on the planar wave are similar, but kernels K_b and K_c derived here actually generalize those for subharmonic resonance in that the former reduce to the latter in the special case $\hat{\sigma}=\hat{\sigma}_d=1/2$.

The appropriate initial condition of (5.11) is determined by the requirement that the solution in the present stage matches to that in stage II. After rewriting (4.2) and $B^\dagger(x^\dagger)$ in terms of $\tilde{x}$, we have that

$$\tilde{A}(\tilde{x}) \to \bar{A}(x_s^\dagger)\exp(\Phi'(x_s^\dagger)\tilde{x}), \quad \tilde{B}(\tilde{x}) \to \exp(\kappa_b x_s^\dagger) \quad \text{as} \quad \tilde{x}\to -\infty. \tag{5.18}$$

The normal distribution of the leading–order streamwise and spanwise velocities continues to undergo deformation, now under the combined influence of both the phase-locked interaction and the self-interaction. More importantly, the distortion of the modal shape takes place over the same length scale as the amplitudes.

5.2. *Study of the amplitude equations*

The amplitude equations (5.11) and (5.12) subject to the initial conditions (5.18) are solved numerically by using an Adam–Moulton finite-difference scheme. In order to march the solution downstream, we assume that when $\tilde{x} \leqslant -T_0 \ll 0$, $\tilde{A}$ and $\tilde{B}$ can be approximated by its asymptote, i.e. by the right–hand side of (5.18). The integrals over the infinite domain $D=[0,\infty)\times[0,\infty)$ are evaluated by integrating over a sufficiently large but finite domain, $D_0=[0,X_0]\times[0,Y_0]$ say, while the tails over $(D-D_0)$ are approximated by using the asymptotes of $\tilde{A}$ and $\tilde{B}$.

The problem involves a number of parameters: γ, θ, Δ, $x_s^\dagger$, A_0 and B_0. In our calculations, we choose the first three to be the same as those in figure 2 so as to examine the continuation of the solutions in the previous stage into the present fully interactive stage. The value of B_0 is taken to be unity without losing generality. As a representative case, we choose $\kappa_b x_s^\dagger=4$ and $A_0=1$.

Numerical solutions suggest that the solution for $\tilde{A}(\tilde{x})$ and $\tilde{B}(\tilde{x})$ terminates at a singularity at a finite distance, $\tilde{x}_s$ say. The singularity has the same structure as that proposed by Wu & Stewart (1996) for the phase-locked interaction of Rayleigh waves,

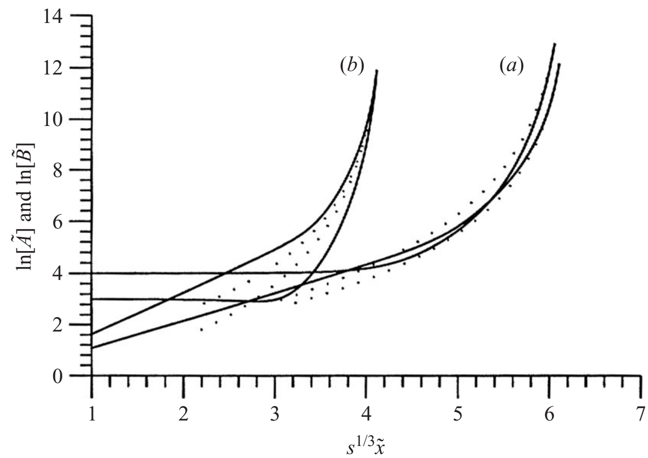

FIGURE 4. $\ln|\tilde{A}|$ and $\ln|\tilde{B}|$ *vs.* $s^{1/3}\tilde{x}$ for $\gamma = 0.12$: (*a*) $\theta = 40°$, (*b*) $\theta = 70°$. The dotted lines represent the local singular solution (5.19).

namely,

$$\tilde{A}(\tilde{x}) \rightarrow \frac{\tilde{a}_0}{(\tilde{x}_s - \tilde{x})^{3+\mathrm{i}\psi_a}}, \quad \tilde{B}(\tilde{x}) \rightarrow \frac{\tilde{b}_0}{(\tilde{x}_s - \tilde{x})^{7/2+\mathrm{i}\psi_b}} \quad \text{as} \quad \tilde{x} \rightarrow \tilde{x}_s, \tag{5.19}$$

where $\tilde{a}_0$ and $\tilde{b}_0$ are complex, and ψ_a and ψ_b are real numbers, respectively. The values of ψ_a and ψ_b along with $|\tilde{a}_0|$ and $|\tilde{b}_0|$ can be determined analytically (the details can be found in Wu & Stewart 1996). It is worth noting that while the singularity in $\tilde{A}$ is of the same form as in the cases of oblique-mode interaction (Goldstein & Choi 1989; Wu *et al.* 1997) and subharmonic resonance (Goldstein & Lee 1992), the singularity in $\tilde{B}$ is different because of the cubic kernel K_p in (5.11).

The development of the amplitudes is shown in figure 4. The oblique modes first induce a back reaction on the planar mode, causing the latter to amplify rapidly. The subsequent two-way interactions lead to the formation of the singularity.

From the physical point of view, the most important feature of the present regime is the rapid distortion of the modal shape with the critical layer. This is illustrated in figure 5. For $\theta = 40°$, the deformation starts with a drifting of the position of the maximum velocity away from the wall (figure 5*a*), while for $\theta = 70°$, the position of the maximum descends towards the wall (figure 5*b*). In both cases, the velocity distribution evolves into a much flatter pattern in the later stage, indicating that the disturbance is no longer concentrated in a thin layer. The thickening of the critical layer and the deformation of $\tilde{U}_1$ and $\tilde{W}_1$ can be described by a similarity variable (Wu *et al.* 1997)

$$\hat{\eta} = (\tilde{x}_s - \tilde{x})\eta. \tag{5.20}$$

The function $\tilde{\Pi}$ defined in (5.10), (which describes the normal distribution of the leading-order streamwise and spanwise velocity components of the oblique modes), has the asymptotic solution

$$\tilde{\Pi} \rightarrow (\tilde{x}_s - \tilde{x})^{-(2+\mathrm{i}\psi_a)} \int_0^\infty (1+\xi)^{-(3+\mathrm{i}\psi_a)} \exp(-\mathrm{i}\alpha_0 \lambda \hat{\eta} \xi)\, \mathrm{d}\xi. \tag{5.21}$$

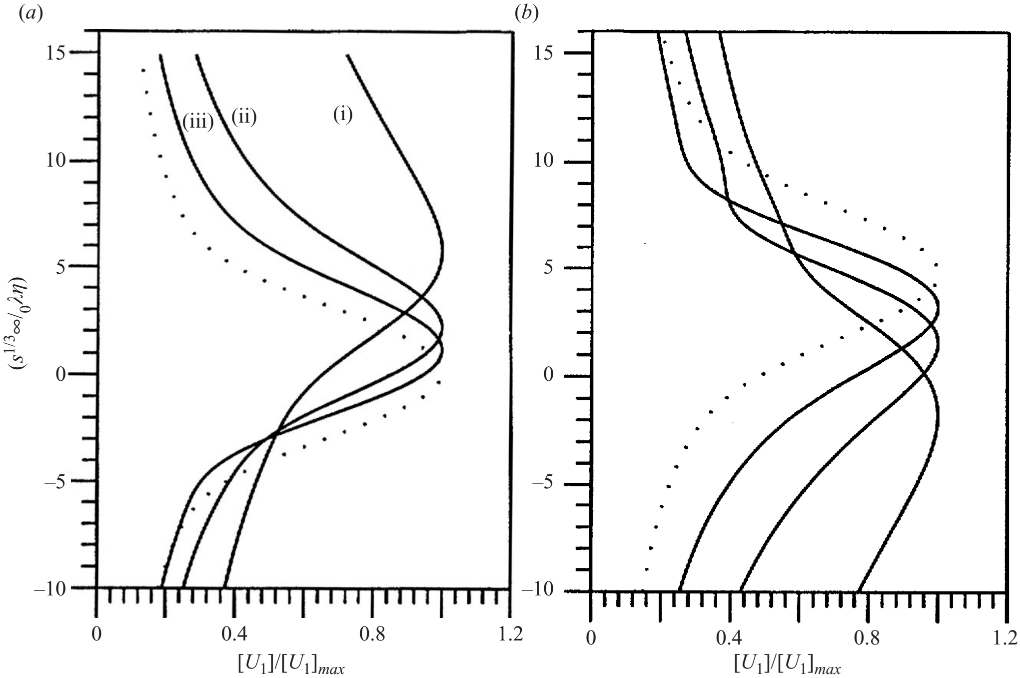

FIGURE 5. 'Shape deformation' in the fully interactive stage illustrated by the modulus of streamwise velocity, $|\tilde{U}_1|$, at different downstream locations. (*a*) $\theta = 40°$, $s^{1/3}\tilde{x} = 4.13$ (i), 4.63 (ii), 5.13 (iii). (*b*) $\theta = 70°$, $s^{1/3}\tilde{x} = 3.5$ (i), 3.75 (ii), 4.0 (iii). The dotted line represents the distribution in the upstream limit $\tilde{x} \to \infty$.

In figure 6, we plot the distributions of $|\tilde{U}_1|$ at different streamwise stations against $\hat{\eta}$. The shape of $\tilde{U}_1$ appears to approach the final similarity form (5.21) as $\tilde{x} \to \tilde{x}_s$, confirming the asymptotic result.

The spanwise dependent mean-flow distortion generated by the oblique-wave interaction is reduced to zero in a viscous wall layer. Its width shrinks as $\sigma^{7/3}(\tilde{x}_s - \tilde{x})^{1/3}$ when $\tilde{x}_s$ is approached. This layer remains passive although it becomes strongly nonlinear when $(\tilde{x} - \tilde{x}_s) = \sigma^{7/5}$, at which stage its dynamics is governed by the interactive boundary-layer equations with known displacement but an unknown spanwise pressure gradient. In this 'inverse problem' (cf. Smith & Daniels 1981), the pressure gradient is allowed to adjust itself. On this basis, we will assume that no singularity forms prior to $\tilde{x}_s$, and the discussion in the next section will be based on this assumption, the confirmation of which however, requires a numerical solution of the interactive boundary-layer equations.

6. Nonlinear stage IV: unsteady inviscid triple-deck regime

As was noted in Wu *et al.* (1997), with the critical-layer thickening like $(\tilde{x}_s - \tilde{x})^{-1}$ (see (5.20)), it eventually merges with the Tollmien layer when

$$\tilde{x}_s - \tilde{x} = O(\sigma^2). \tag{6.1}$$

This and ensuing scenarios of the evolution are similar to those for subharmonic resonance in a decelerating boundary layer, for which the critical layer is non-equilibrium in the linear regime (Goldstein & Lee 1992). In the neighbourhood

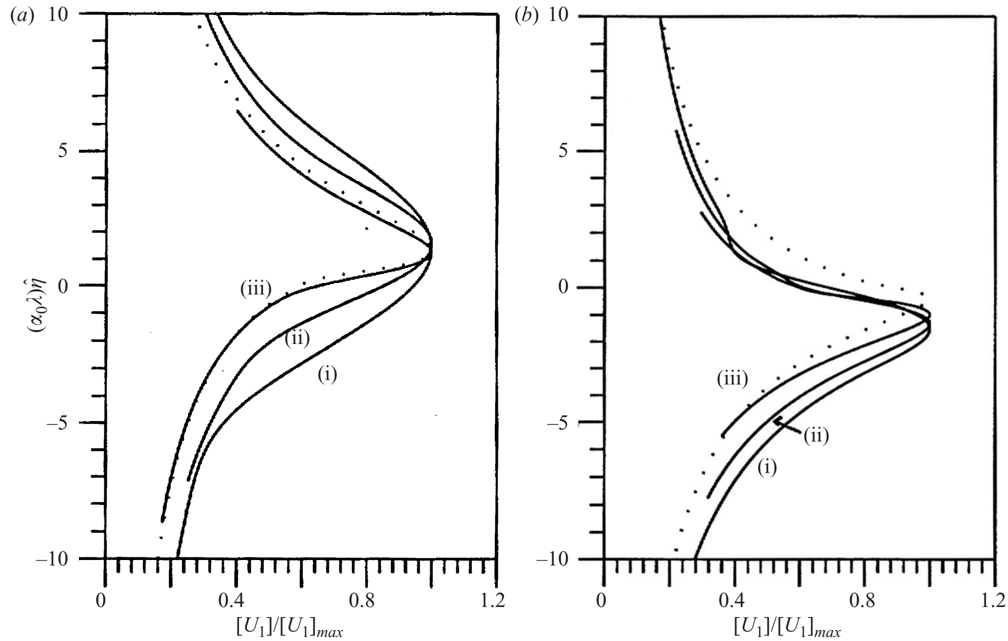

FIGURE 6. Distribution of $|\tilde{U}_1|$ in terms of the similarity variable $\hat{\eta}$. (*a*) $\theta = 40°$, $s^{1/3}\tilde{x} = 4.13$ (i), 4.63 (ii), 5.13 (iii). (*b*) $\theta = 70°$, $s^{1/3}\tilde{x} = 3.75$ (i), 4.0 (ii), 4.125 (iii). The dotted line corresponds to the asymptotic form (5.21).

specified by (6.1), the disturbance enters yet another new stage which is distinguished in that (*a*) the unscaled growth rates of both the planar and oblique waves, $\sigma^3(\tilde{B}'/\tilde{B})$ and $\sigma^3(\tilde{A}'/\tilde{A})$, have now increased to $O(\sigma)$ and are of the same order as their wavenumbers, and (*b*) all the harmonics have the same order of magnitude as that of the fundamental. Note also that the planar mode has acquired the same size as the oblique modes. The solution in this regime can be described by the streamwise variable

$$\bar{x} = c\,(\tilde{x} - \tilde{x}_s)/\sigma^2. \tag{6.2}$$

along with the time and spanwise variables

$$\bar{t} = \sigma^2 t, \quad \bar{z} = \sigma z.$$

The flow now consists of four layers: the upper layer, the main layer, the Tollmien layer and a viscous wall sublayer. It turns out that the flow is governed by the inviscid version of the nonlinear three-dimensional unsteady triple-deck equations. It has been shown earlier that the nonlinear interaction between a pair of oblique T-S modes or among a resonant triad can evolve into such a regime (Wu *et al.* 1997; Goldstein 1994, 1995a). The Tollmien layer, which is described by the transverse variable

$$Y = y/\sigma,$$

becomes fully nonlinear, and the solution in this layer expands as

$$(u, v, w, p) = (\sigma U, \sigma^3 V, \sigma W, \sigma^2 P) + \cdots, \tag{6.3}$$

as suggested by (5.6)–(5.8), (5.9), (5.10), (5.21) and (6.2). Substitution of the above expansion into the Navier–Stokes equations yields, to leading order, the governing equations for U, V, W and P

$$\left.\begin{aligned}\frac{\partial U}{\partial \bar{x}}+\frac{\partial V}{\partial Y}+\frac{\partial W}{\partial \bar{z}}&=0,\\ \frac{\partial U}{\partial \bar{t}}+U\frac{\partial U}{\partial \bar{x}}+V\frac{\partial U}{\partial Y}+W\frac{\partial U}{\partial \bar{z}}&=-\frac{\partial P}{\partial \bar{x}},\\ \frac{\partial W}{\partial \bar{t}}+U\frac{\partial W}{\partial \bar{x}}+V\frac{\partial W}{\partial Y}+W\frac{\partial W}{\partial \bar{z}}&=-\frac{\partial P}{\partial \bar{z}}.\end{aligned}\right\} \tag{6.4}$$

The pressure P, which is a function of $\bar{x}$, $\bar{t}$ and $\bar{z}$ only, is related to the displacement $D(\bar{x},\bar{t},\bar{z})$ via (see e.g. Smith & Stewart 1987; Zhuk & Ryzhov 1989)

$$P=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 D/\partial\xi^2}{[(\bar{x}-\xi)^2+(\bar{z}-\eta)^2]^{1/2}}\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{6.5}$$

The system (6.4)–(6.5) is subjected to the boundary condition: $V=0$ at $Y=0$, while the condition as $Y\to+\infty$ is given by matching U and W with their counterparts in the main layer, namely (Zhuk & Ryzhov 1989; Wu *et al.* 1997)

$$\left.\begin{aligned}U&\to\lambda Y+D(\bar{x},\bar{t},\bar{z})+\left\{\int_0^{\infty}\xi P_{\bar{z}\bar{z}}(\bar{x}-\xi,\bar{t},\bar{z})\,\mathrm{d}\xi\right\}Y^{-1},\\ W&\to\left\{\int_0^{\infty}P_{\bar{z}}(\bar{x}-\xi,\bar{t},\bar{z})\,\mathrm{d}\xi\right\}Y^{-1}.\end{aligned}\right\} \tag{6.6}$$

Since the present stage follows from the formation of the singularity (5.19), the appropriate 'initial condition' is provided by matching with the asymptotic behaviour of the critical-layer solution in the previous stage. After rewriting (5.6)–(5.8) in terms of (6.2) by making use of (5.9), (5.10), (5.19)–(5.21), we find that as $\bar{x}\to-\infty$,

$$\begin{aligned}\begin{bmatrix}U-\lambda Y\\ V\\ W\\ P\end{bmatrix}\to&\begin{bmatrix}\left(-\mathrm{i}\lambda\gamma\sin^2\theta\,\bar{x}G(\bar{x},Y)+\lambda\sec\theta\right)\cos\beta\bar{z}\\ (-\mathrm{i}\gamma\lambda)Y\cos\beta\bar{z}\\ -\lambda\gamma\sin\theta\cos\theta\,\bar{x}G(\bar{x},Y)\sin\beta\bar{z}\\ \gamma\cos\theta\cos\beta\bar{z}\end{bmatrix}\\ &\times\tilde{a}_0(-\bar{x}/c_0)^{-(3+\mathrm{i}\psi_a)}\exp(\mathrm{i}\alpha\phi+\mathrm{i}\alpha(\bar{x}-c\bar{t}))\\ &+\begin{bmatrix}\lambda\\ (-\mathrm{i}\tilde{\alpha}_0\lambda Y)\\ 0\\ \tilde{\alpha}_0\end{bmatrix}\tilde{b}_0(-\bar{x}/c_0)^{-(7/2+\mathrm{i}\psi_b)}\exp(\mathrm{i}\tilde{\alpha}\phi+\mathrm{i}\tilde{\alpha}[\bar{x}-(c+\sigma^2\Delta)\bar{t}\,])+\mathrm{c.c.},\end{aligned} \tag{6.7}$$

where

$$G(\bar{x},Y)=\int_0^{\infty}(1+\xi)^{-(3+\mathrm{i}\psi_a)}\exp(\mathrm{i}\alpha_0\lambda\bar{x}(Y-Y_c)\xi)\,\mathrm{d}\xi, \tag{6.8}$$

which is obtained by rewriting (5.21) in terms of $\bar{x}$ and Y, while

$$\phi=c_0(x_s^{\dagger}+\kappa_b^{-1}\log\sigma^{-1/2}+\sigma\tilde{x}_s)/\sigma^3.$$

The system (6.4)–(6.6) was derived by Zhuk & Ryzhov (1989). The present study suggests that such a regime may indeed be reached from a well-defined initially linear stage, via a sequence of intermediate weakly nonlinear stages (i.e. stages I–III). Moreover, by following through these, we are able to give the appropriate initial condition that this system must satisfy. The present study and that of Wu *et al.* (1997) indicate that the inviscid three-dimensional triple-deck equations are relevant for the late stage of transition initiated by small disturbances upstream.

Finally, it should be noted that a wall sublayer with a width of $O(\sigma^4)$ has to be introduced in order to satisfy the no-slip condition on the wall. The transverse variable is $\tilde{Y} = y/\sigma^4$ and the expansion takes the form

$$(u, v, w) = (\sigma \tilde{U}, \sigma^6 \tilde{V}, \sigma \tilde{W}) + \cdots . \tag{6.9}$$

Substitution of this into the Navier-Stokes equations shows that the flow field $(\tilde{U}, \tilde{V}, \tilde{W})$ is governed by the unsteady, classical boundary-layer equations

$$\left.\begin{aligned} \frac{\partial \tilde{U}}{\partial \bar{x}} + \frac{\partial \tilde{V}}{\partial \tilde{Y}} + \frac{\partial \tilde{W}}{\partial \bar{z}} &= 0, \\ \frac{\partial \tilde{U}}{\partial \bar{t}} + \tilde{U}\frac{\partial \tilde{U}}{\partial \bar{x}} + \tilde{V}\frac{\partial \tilde{U}}{\partial \tilde{Y}} + \tilde{W}\frac{\partial \tilde{U}}{\partial \bar{z}} &= -\frac{\partial P}{\partial \bar{x}} + \frac{\partial^2 \tilde{U}}{\partial \tilde{Y}^2}, \\ \frac{\partial \tilde{W}}{\partial \bar{t}} + \tilde{U}\frac{\partial \tilde{W}}{\partial \bar{x}} + \tilde{V}\frac{\partial \tilde{W}}{\partial \tilde{Y}} + \tilde{W}\frac{\partial \tilde{W}}{\partial \bar{z}} &= -\frac{\partial P}{\partial \bar{z}} + \frac{\partial^2 \tilde{W}}{\partial \tilde{Y}^2}, \end{aligned}\right\} \tag{6.10}$$

with P being already given once the inviscid problem (6.4)–(6.5) is solved.

Solutions of the inviscid triple-deck equations are likely to develop singularities, and it is known that solutions of the unsteady classical boundary-layer equation (6.10) can develop finite-time 'separation' singularities (e.g. Van Dommelen & Shen 1980; Van Dommelen & Cowley 1990). It is highly probable that there will be a further cascade of shorter scales generated in this regime as well, although we do not pursue that matter here.

7. Discussion and conclusion

In this paper, we have followed the nonlinear evolution of phase-locked planar and oblique T-S waves, and presented a self–consistent asymptotic description of the nonlinear stages through which this form of disturbance evolves. We show that for perfectly phase-locked modes, the initial nonlinear interaction simply alters the wavelength of the oblique modes, without affecting their magnitude. However, once the wavelength alteration becomes sufficiently rapid, it produces a back reaction on the magnitude, causing the latter to amplify super-exponentially. If a small but non-zero mismatch exists between the phase speeds, the interaction induces a super-exponential growth/decay immediately. For any given planar mode $(\tilde{\alpha}, 0)$ and oblique mode (α, β) (with $\alpha^2 + \beta^2 = \tilde{\alpha}^2$ to leading order), there is an 'optimal' phase-speed mismatch which gives the maximum rate of super-exponential growth. Whether or not the phase-speed mismatch is zero, the rapid amplification of the oblique modes leads to the second stage where the critical layer is both viscous and non-equilibrium in its nature. An important feature of the new regime is that the 'shape' of the oblique modes deforms during the evolution. The planar mode still evolves exponentially, but the oblique modes experience a super-exponential growth of a different form from that in the previous stage. The disturbance enters the third stage when the self-interaction between the oblique modes and their back effect on the planar mode become important. The amplitude equations of this fully interactive regime are found to consist of the same nonlinear terms as those derived by Wu & Stewart (1996) for the phase-locked interaction of Rayleigh instability waves (and they also bear some resemblance to the evolution equations for the subharmonic resonance, Goldstein 1995*a*; Wu 1995). Since the linear dynamics associated with viscosity becomes secondary, it may be said that the Tollmien–Schlichting waves now take on the characteristics of Rayleigh waves. The 'shape' of the oblique modes undergoes further deformation over the same length scale as their amplitude. The solution to the amplitude equations is found to develop a singularity within a finite distance for the parameters considered. We expect this to be a generic feature of the amplitude equations. In the vicinity of the singularity, the flow is governed by the fully nonlinear three-dimensional inviscid triple-deck equations. Note that a similar evolution process, that is the development from the viscous critical-layer regime to the non-equilibrium critical-layer regime and finally to the fully nonlinear triple-deck stage, also takes place for the disturbance in the form of subharmonic resonant triad (Goldstein 1994, 1995*a*) or of a pair of oblique modes only (Wu *et al.* 1997). This final regime serves as the 'attracting stage' of three potentially important forms of disturbances.

In the Appendix, we further show that the amplitude equations that govern the weakly nonlinear stages can be obtained as limiting forms of the amplitude equations (5.34)–(5.35) in Wu & Stewart (1996), which were derived in the non-equilibrium, viscous nonlinear critical layer regime. The latter therefore provide a uniformly valid description of the whole linear and weakly nonlinear evolution process. This result implies that one may use them as 'composite' equations without tracing various stages, which may, from the practical point of view, be more convenient. Thus (5.34)–(5.35) in Wu & Stewart (1996) are the generic equations governing the weakly nonlinear development of phase-locked modes of both T-S and Rayleigh waves, in pretty much the same way as the amplitude equations (4.8)–(4.9) in Wu (1995) describe the weakly nonlinear evolution of subharmonic resonant triads of both T-S and Rayleight waves (Goldstein 1995*a*).

A main result of our study is that through the phase-locked interaction, the dominant planar T-S wave would induce rapid growth of certain oblique modes while its own nonlinear effect is largely negligible. In this sense, the planar mode acts as a catalyst, as in the subharmonic resonance mechanism. Of significance is that the phase-locked interaction operates at a much less restrictive condition. The theoretical result implies that introduction of a single planar mode to the boundary can promote all three-dimensional disturbances that share approximately the same phase speed. This mechanism may have important implications for flow control. We suggest that it may be responsible for the appearance of broadband disturbances in the late stage of boundary-layer transition. Strong evidence for such a catalytic effect through phase-locked interaction has been found in the experiments of Borodulin, Kachanov & Koptsev (2002*a*–*c*), in which broadband low-amplitude noise-like three-dimensional T-S waves are present among the background disturbances. In the unforced case, these components simply exhibit a slow growth as anticipated by linear theory. However, when a single two-dimensional primary wave was introduced, three-dimensional waves in a wide continuous frequency range all underwent rapid amplification in the form of super-exponential growth to acquire amplitudes which were at least one order-of-magnitude larger than those in the unforced case; see figures 13, 14 and 18 of their paper. All these components were phase-locked (synchronized) with the seeded planar wave in that they all had nearly the same phase speed, as shown in their figure 20, which we reproduce in figure 7 for convenience Although such a striking effect was observed in a decelerating boundary layer, we believe that it is due to the phase-locked interaction described in this paper because the essential mechanism is generic and is little affected by the presence of an adverse pressure gradient. In fact, for a suitable

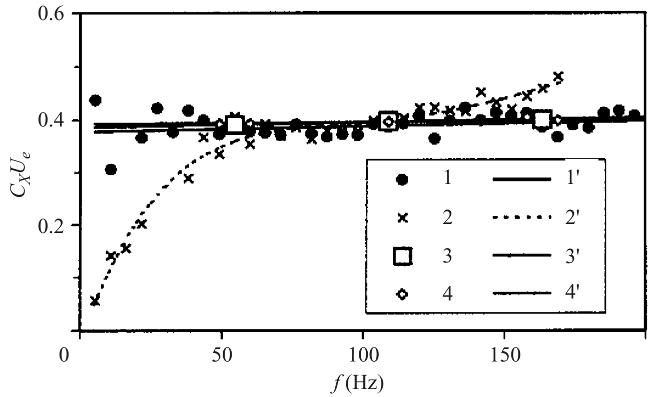

FIGURE 7. Phase-speed C_X (normalized by the slip velocity U_e) *vs.* frequency f, showing phase synchronism of amplified broadband disturbances. Forced case: ● 1 —1′; unforced case: ×, 2 ··· 2′. [Reproduced from Borodulin, Kachanov & Koptsev 2002.]

small adverse pressure gradient, the critical layer can be non-equilibrium at the onset (cf. Goldsetein & Lee 1992) so that the amplitude equations derived in Wu & Stewart (1997), as well as the associated properties, are directly applicable.

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A. Appendix

The nonlinear, non-equilibrium, viscous critical-layer amplitude equations of Wu & Stewart (1996) can be written as

$$\frac{\mathrm{d}A}{\mathrm{d}x_1} = \kappa_a A + \mathrm{i}\Upsilon_p \int_0^\infty \int_0^\infty K_p(\xi,\eta|\Lambda)\exp\left(-\mathrm{i}\sigma^{-1}\alpha_0\Delta(\xi+\eta)\right)B(x_1-\hat{\sigma}_d\xi) \times B^*(x_1-\xi-\hat{\sigma}\eta)A(x_1-\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \mathrm{i}\Upsilon_a \int_0^\infty \int_0^\infty K_a(\xi,\eta|\Lambda)A(x_1-\xi)A(x_1-\xi-\eta)A^*(x_1-2\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \quad \text{(A 1)}$$

$$\frac{\mathrm{d}B}{\mathrm{d}x_1} = \kappa_b B + \mathrm{i}\Upsilon_b \int_0^\infty \int_0^\infty K_b(\xi,\eta|\Lambda)\exp\left(-\mathrm{i}\sigma^{-1}\tilde{\alpha}_0\Delta(\xi+\eta)\right)A(x_1-\xi) \times B(x_1-\xi-\eta)A^*(x_1-\nu_s\xi-\nu_0\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \mathrm{i}\Upsilon_c \int_0^\infty \int_0^\infty K_c(\xi,\eta|\Lambda)\exp(-\mathrm{i}\sigma^{-1}\tilde{\alpha}_0\Delta\xi) \times B(x_1-\xi)A(x_1-\xi-\eta)A^*(x_1-\nu_s\xi-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \quad \text{(A 2)}$$

where Λ is the viscous parameter. The coefficients of the nonlinear terms, Γ_p and Γ_a etc., are complex constants in general, but are real in the long-wavelength limit. For the Blasius boundary layer of interest here, the viscous parameter $\Lambda = s\sigma^{-3}$. If the magnitude of the planar wave is scaled as in § 3 (i.e. $\epsilon = \sigma^{17/2}$), and that of the oblique modes $\delta = \sigma^7$, then Γ_p and Γ_a etc. are given by those in (5.14)–(5.16) divided by σ^6, that is $\Gamma_p = \sigma^{-6}\tilde{\Gamma}_p$ etc. Equations (A 1)–(A 2) can be derived if the formally small term $\partial/\partial x_1$ is retained at the leading order in the critical-layer equations. Alternatively, they may be obtained by retaining the formally small linear-growth terms in (5.11)–(5.12), and then performing appropriate rescaling.

In the following, we show that the various evolution equations derived in §§ 3–5, which govern the three weakly nonlinear stages of the phase-locked T-S waves, are all contained within (A 1)–(A 2) in the sense that the former equations may be derived as the limiting forms of the latter. In the rest of this Appendix up to the second last paragraph, we assume that $A \ll O(1)$ such that the second nonlinear term in (A 1) as well as the nonlinear terms in (A 2) can be neglected and that $B = B_0 \exp(\kappa_b x_1)$.

The amplitude equation, (3.26), in the first nonlinear stage can immediately be obtained by substituting

$$\xi = \sigma \bar{\xi}, \quad \eta = \sigma \bar{\eta} \tag{A 3}$$

into (A 1) and then taking the limit $\sigma \to 0$.

In the case of zero phase-speed mismatching (i.e. $\Delta = 0$), we may introduce (cf. Goldstein 1994)

$$x_1 = \hat{x} + \kappa_b^{-1} \log \sigma^{-1/4}, \quad A = \hat{A}(\hat{x}) \exp\left(\mathrm{i}\hat{\Theta}/\sigma^{1/2}\right). \tag{A 4}$$

with $\hat{x}$ being of order one. Inserting (A 4) and (A 3) along with $B = \exp(\kappa_b x_1) = \sigma^{-1/4}\hat{B}(\hat{x})$ into (A 1) yields

$$\left\{\hat{A}' + \mathrm{i}\sigma^{-1/2}\hat{\Theta}'\hat{A}\right\} \exp\left(\mathrm{i}\hat{\Theta}/\sigma^{1/2}\right) = \kappa_a \hat{A} \exp\left(\mathrm{i}\hat{\Theta}/\sigma^{1/2}\right) + \sigma^{-1/2}\mathrm{i}\tilde{\Upsilon}_p |\hat{B}|^2 \hat{N}(\hat{x}), \tag{A 5}$$

where

$$\begin{aligned}\hat{N}(\hat{x}) = &\int_0^\infty \int_0^\infty K_p(\xi, \eta | s) \exp(-\sigma\kappa_b[(1+\hat{\sigma}_d)\xi + \hat{\sigma}\eta] + \sigma^{-1/2}\hat{\Theta}(\hat{x} - \sigma(\xi+\eta))) \\ &\times \hat{A}(\hat{x} - \sigma(\xi+\eta))\, \mathrm{d}\xi\, \mathrm{d}\eta \\ &\to \left\{I_2^2(0)\hat{A} - (2\mathrm{i}I_2(0)I_3(0)\hat{\Theta}')\sigma^{1/2}\hat{A} + O(\sigma)\right\} \exp(\mathrm{i}\hat{\Theta}/\sigma^{1/2});\end{aligned} \tag{A 6}$$

here in the last step, we approximated the integrand by its Taylor series for small σ and then took the limit $\sigma \to 0$. Equating the terms at $O(\sigma^{-1/2})$ and $O(1)$ in (A 5), we obtain the equations

$$\hat{\Theta}' = \mathrm{i}\tilde{\Upsilon}_p I_2^2(0)|\hat{B}|^2, \quad \hat{A}' = \kappa_a \hat{A} + 2\tilde{\Upsilon}_p I_2(0) I_3(0) \hat{\Theta}' |\hat{B}|^2 \hat{A}.$$

These are identical to (3.43) and (3.57) with the identical coefficients.

In the non-equilibrium WKBJ stage, we introduce

$$x_1 = x^\dagger + \kappa_b^{-1} \log \sigma^{-1/2}, \quad A = \hat{A}(x^\dagger) \exp(\Phi(x^\dagger)/\sigma). \tag{A 7}$$

Substitution of these and (A 3) into (A 1) gives

$$\{\hat{A}' + \sigma^{-1}\Phi'\hat{A}\} \exp(\Phi/\sigma) = \kappa_a \hat{A} \exp(\Phi/\sigma) + \sigma^{-1}\mathrm{i}\tilde{\Upsilon} \exp(2\kappa_b x^\dagger) N_p \tag{A 8}$$

where

$$\begin{aligned}N_p = &\int_0^\infty \int_0^\infty K_p(\xi, \eta | s) \exp(-\sigma\kappa_b[(1+\hat{\sigma}_d)\xi + \hat{\sigma}\eta] - \mathrm{i}\alpha_0 \Delta(\xi+\eta) + \sigma^{-1}\Phi(x^\dagger - \sigma(\xi+\eta))) \\ &\times \bar{A}(x^\dagger - \sigma(\xi+\eta))\, \mathrm{d}\xi\, \mathrm{d}\eta.\end{aligned}$$

A Taylor expansion of the integrand shows that

$$\begin{aligned}N_p \to\ &\bar{A}\left\{\int_0^\infty \xi^2 \exp(-s\hat{\sigma}_d \xi^3 - \mathrm{i}\alpha_0 \Delta\xi - \Phi'(x^\dagger)\xi)\, \mathrm{d}\xi\right\}^2 \exp(\Phi/\sigma) \\ &+ \sigma\, \mathrm{e}^{\Phi/\sigma} \int_0^\infty \int_0^\infty K_p(\xi, \eta | s) \exp(-(\xi+\eta)(\mathrm{i}\alpha_0 \Delta + \Phi')) \\ &\times \left\{\tfrac{1}{2}(\xi+\eta)^2 \Phi'' \bar{A} - \kappa_b[(1+\hat{\sigma}_d)\xi + \hat{\sigma}\eta]\bar{A} - (\xi+\eta)\bar{A}'\right\} \mathrm{d}\xi\, \mathrm{d}\eta.\end{aligned} \tag{A 9}$$

The balance in (A 8) at $O(\sigma^{-1})$ gives

$$\Phi'(x^{\dagger}) = \mathrm{i}\tilde{\Upsilon}_p \exp(2\kappa_b x^{\dagger}) \left\{ \int_0^{\infty} \xi^2 \exp(-s\hat{\sigma}_d \xi^3 - \mathrm{i}\alpha_0 \Delta\xi - \Phi'(x^{\dagger})\xi)\, \mathrm{d}\xi \right\}^2, \qquad (\mathrm{A}\,10)$$

which is equivalent to (4.17). The balance at $O(1)$ gives the equation for $\bar{A}$, which, after using the relations obtained by differentiating (A 10), can be simplified to

$$\bar{A}' = \frac{1}{2} \left\{ \frac{\kappa_a \Phi''}{\kappa_b \Phi'} - \frac{\Phi''}{\Phi'} \left(\frac{\Phi'}{\Phi''} \right)' \right\} \bar{A}. \qquad (\mathrm{A}\,11)$$

The solution for $\bar{A}$ is then

$$\bar{A} = \bar{a}_0 (\Phi')^{(\kappa_a - \kappa_b)/(2\kappa_b)} (\Phi'')^{1/2}, \qquad (\mathrm{A}\,12)$$

with $\bar{a}_0$ being a constant. This is the equation that would be obtained if the analysis in §4 is carried to higher orders.

In the final weakly nonlinear regime, the disturbance is described by the faster variable $\tilde{x}$, which is related to x_1 by

$$x_1 = \sigma\tilde{x} + x_s^{\dagger} + \kappa_b^{-1} \log \sigma^{-1/2}.$$

Substituting this and (A 3) along with $B = \sigma^{-1/2}\tilde{B}$ into (A 1)–(A 2), we obtain (5.11)–(5.12).

The result of this Appendix therefore shows that the amplitude equations (A 1)–(A 2) provide a uniformly valid description of the evolution of the phase-locked T–S waves throughout the entire linear and weakly nonlinear regimes.

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